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The selection of Saffman-Taylor fingers by kinetic undercooling

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Abstract. The selection of Saffman-Taylor fingers by surface tension has been widely studied. Here their selection is analysed by another regularisation widely adopted in studying otherwise ill-posed Stefan problems, namely kinetic undercooling. An asymptotic-beyond-all-orders analysis (which forms the core of the paper) reveals for small kinetic undercooling how a discrete family of fingers is selected; while these are similar to those arising for surface tension, the asymptotic analysis exhibits a number of additional subtleties. In Appendix 1 a description of some general features of the Hele-Shaw problem with kinetic undercooling and an analysis of the converse limit in which kinetic undercooling effects are large are included, while Appendix 2 studies the role of exponentially small terms in a simple linear problem which clarifies the rather curious behaviour at the origin of Stokes lines in the Hele-Shaw problem with kinetic undercooling.

Key words: asymptotics, beyond-all-orders, finger selection, Hele-Shaw, kinetic undercooling

1. Introduction

In 1958 Saffman and Taylor [1] demonstrated a family of solutions to the Hele-Shaw equations which describe a finger of inviscid fluid displacing a viscous fluid in a Hele-Shaw channel, with the relative width λ of the inviscid finger being arbitrary. In experiments [1] they found that λ was always close to 1/2 and that the experimental finger profile was very close to the theoretical profile for $\lambda = 1/2$.

A constant pressure condition was adopted on the moving boundary in [1] and it was conjectured that the presence of a small surface tension would select the solution $\lambda = 1/2$ from the continuum, but a naive perturbation expansion in powers of the surface tension still allowed arbitrary λ. Several authors then showed that terms which are exponentially small in the surface tension are responsible for the selection of λ [2–6] in the presence of small surface tension, giving a discrete spectrum of admissible values. A number of other regularisations (such as anisotropic surface tension effects, non-parallel gap effects, contact angle effects) have also been studied (see [7] and references therein) in terms of how possible values of λ are selected via terms which are exponentially small in the regularising parameter; other authors (*e.g.* [8, 9]) have sought to develop selection criteria which are not regularisation dependent. With regard to the former, we emphasise that the current study leads to beyond-all-orders subtleties not present in existing analyses while, with respect to the latter, we believe it also goes some way towards clarifying how the exceptional status of the complex plane structure of the case $\lambda = 1/2$ (specifically, its coincident complex plane singularities) leads to its being selected by a wide variety of different regularisations, including kinetic undercooling.

The Hele-Shaw problem also arises as the large-latent-heat (small-specific-heat) limit of the Stefan problem for solid/liquid phase changes. The physical effect most widely incorporated in regularising the ill-posed Stefan problem, apart from surface energy, is kinetic undercooling, whereby the value of the dependent variable (the temperature) on the moving boundary depends on the normal velocity of that boundary (see, for example, [10–12] for relevant background and other references). The Stefan problem also arises in the theory of superconductivity as a model for the collapse of the superconducting state in the presence of a magnetic field, where it is also regularised by surface tension and kinetic undercooling effects [13]. Equivalent formulations are also relevant to the modelling of epitaxial growth, see [14], for example.

Thus there are very good physical grounds for investigating the role of kinetic undercooling in the Hele-Shaw problem, particularly since there have been few previous studies of such formulations (see [15, 16], however). A discussion of some of the general properties of Hele-Shaw flow with kinetic undercooling, and of the selection problem in the limit complementary to that in the rest of the paper, is relegated to Appendix 1 in order to avoid its interrupting the beyond-all-orders analysis which constitutes the main theme of the paper.

The main aim of the present paper is to show that the inclusion of kinetic undercooling as the regularising mechanism for Saffman-Taylor fingers also leads to the selection of $\lambda = 1/2$ in the limit of small kinetic undercooling parameter, using the method developed recently in [2, 17]; for an alternative approach, see [23], for example.

The presence of singularities in the complex plane of the leading-order solution (*i.e.*, the solution in the absence of kinetic undercooling) leads to a divergent asymptotic expansion in the kinetic undercooling parameter. This expansion exhibits the Stokes phenomenon, whereby exponentially small terms are switched on across certain Stokes lines in the complex plane originating at the singularities of the leading-order solution. The method of [17] is

(i) expand the solution naively as an algebraic asymptotic series;
(ii) apply a WKBJ-type ansatz to the equation for ϕ_n to find the behaviour of ϕ_n as $n \to \infty$;
(iii) truncate the algebraic series optimally and observe the subdominant exponential being switched on across a Stokes line.

By using this method we will be able to see the region in the complex plane in which the subdominant exponentials are present and to deduce the condition for a solution of the regularised problem to exist.

We will find that the structure of the Stokes lines associated with the kinetic undercooling regularisation is novel, so that the problem is of both physical and mathematical interest.

In the next section we formulate the free-boundary problem as a coupled integral and differential equation analagous to that of McLean and Saffman for the Hele-Shaw problem with surface tension [18]. This formulation is convenient for the beyond-all-orders analysis which follows.

In Section 3 we apply the method of [17] to the problem when $\lambda - 1/2 = O(1)$. We develop the algebraic expansion in powers of the kinetic undercooling parameter, determine the behaviour of the late terms up to a constant Λ, and observe the switching on of subdominant exponentials across Stokes lines via a local analysis which smoothes the Stokes discontinuity. We obtain a solvability condition easily and naturally, without the usual need for an inner analysis in the vicinity of the singularity in the complex plane. For completeness, and as a check for consistency, in Section 4 we perform this inner analysis to determine the unknown constant Λ anyway.

We will find that the Stokes lines determined in Section 3 behave rather curiously at the tip of the finger. In Appendix 2 we consider a paradigm problem which clarifies this behaviour.

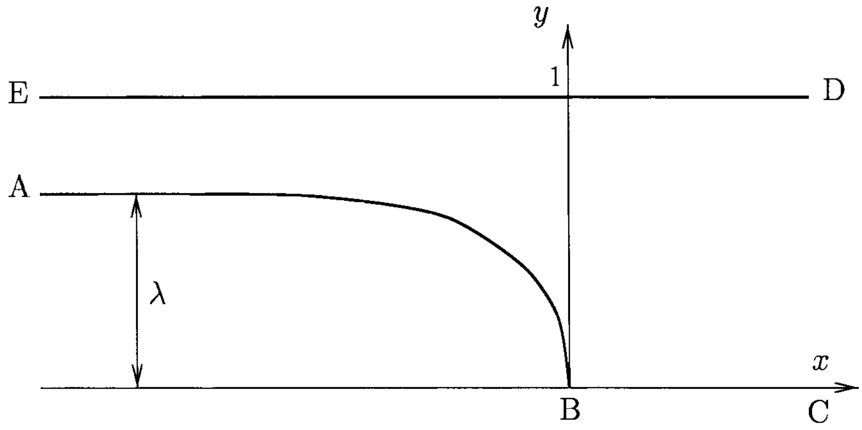


Figure 1. The region in the physical plane showing the upper half of the finger.

For the principal solution branches we find that the solvability condition implies that $\lambda - 1/2$ is in fact small in the limit of small kinetic undercooling. Thus the analysis shows that small kinetic undercooling selects $\lambda = 1/2$. However, to determine how λ approaches 1/2 as the kinetic undercooling parameter tends to zero a separate analysis is required, which we perform in Section 5. In this case matching with an inner solution is required. We find that to determine the solvability condition requires a beyond-all-orders analysis of the nonlinear recurrence relation associated with the inner problem, so that it cannot simply be iterated numerically as usual. This is an extra complication over and above the usual subtleties associated with beyond-all-orders problems, and is beyond the scope of the present paper.

Finally, in Section 7 we present our conclusions.

2. Formulation of the problem

The problem we consider can be thought of either as a Hele-Shaw problem, or as a Stefan problem in the limit of small specific heat, and we start from the nondimensional version of the equations.

The following analysis parallels that in McLean and Saffman for the Hele-Shaw problem with surface tension [18]. We consider a channel of non-dimensional width 2. We suppose that a travelling wave has formed in which a symmetric single finger of width 2λ as $x \to -\infty$ is propagating at a velocity $1/(1-\lambda)$ (see Figure 1). The dimensionless equations are then

$$\nabla^2 \bar{\phi} = 0 \tag{1}$$

outside the finger with

$$\frac{\partial \bar{\phi}}{\partial n} = v_n, \tag{2}$$

$$\bar{\phi} = c v_n, \tag{3}$$

on the free boundary, where $\bar{\phi}$ is the dimensionless velocity potential or temperature, v_n is the normal velocity of the free boundary, and c is the dimensionless kinetic undercooling parameter. Following [1], we work in a frame moving with the finger by setting $\bar{\phi} = x/(1-\lambda) + \phi$. Then

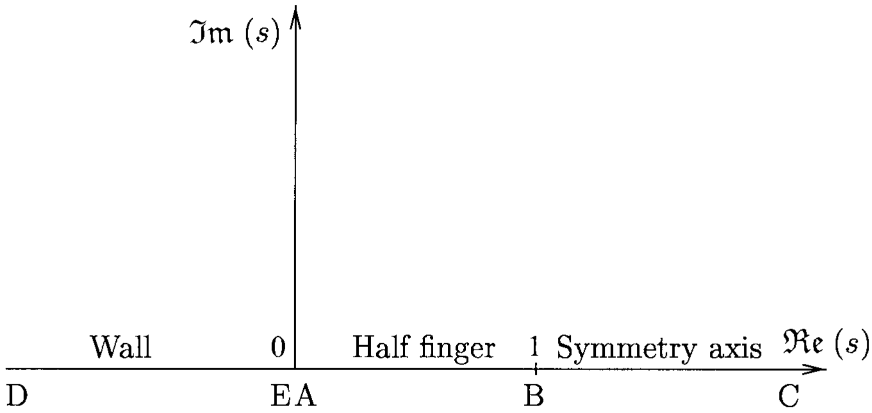


Figure 2. The liquid region in the s plane.

$$\nabla^2\phi = 0, \tag{4}$$

outside the finger with, in the case of travelling wave solutions,

$$\frac{\partial\phi}{\partial n} = 0, \tag{5}$$

$$\phi = cv_n - \frac{x}{1-\lambda}, \tag{6}$$

on the free boundary. On the fixed boundaries $y = \pm 1$ we have

$$\frac{\partial\phi}{\partial y} = 0. \tag{7}$$

The conditions as $x \to \pm\infty$ are

$$\phi \sim -\frac{x}{1-\lambda} \quad \text{as } x \to -\infty, \quad \lambda < |y| < 1, \tag{8}$$

$$\phi \sim -x \quad \text{as } x \to +\infty, \quad -1 < y < 1. \tag{9}$$

As usual, we define the complex potential by $w = \phi + \mathrm{i}\psi$, where ψ is the streamfunction for Hele-Shaw flow, and we let $z = x + \mathrm{i}y$. Then the conformal transformation $z \to w$ maps the liquid region onto an infinite strip of unit width in the potential plane. The second conformal transformation

$$w \to s = \mathrm{e}^{-w\pi} \tag{10}$$

finally maps the liquid region onto the upper half s-plane. The interface AB is mapped to the real segment $0 < s < 1$, with $s = 1$ corresponding to the finger tip. Figure 2 shows the s plane, indicating the positions of various points of interest. It is most convenient to work in terms of the complex velocity

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \hat{u} - \mathrm{i}\hat{v},$$

where $(\hat{u}, \hat{v})$ denotes the velocity. At the interface this velocity must be tangential, and may be written as $\hat{q}\mathrm{e}^{-\mathrm{i}\hat{\theta}}$ where $\hat{\theta}$ is the angle between the tangent and the x direction. Making use of the analyticity of $\log\hat{q} - \mathrm{i}\hat{\theta}$ in the upper half s plane, it can be shown using standard Hilbert transform results that on the interface

$$\log \hat{q} = -\frac{1}{\pi} \fint_0^1 \frac{\hat{\theta}(s') - \pi}{s' - s}\,\mathrm{d}s', \tag{11}$$

with the constants fixed by the condition that $\hat{q} \to 1$ as $\phi \to -\infty$. Equation (11), involving a principal value integral, holds only when s is real. We will need the analytic continuation of (11) into the upper-half complex s plane, which from the Plemelj formulae is

$$\log \hat{q} - \mathrm{i}(\hat{\theta} - \pi) = -\frac{1}{\pi} \int_0^1 \frac{\hat{\theta}(s') - \pi}{s' - s}\,\mathrm{d}s'. \tag{12}$$

Note that the right-hand side of (12) may alternatively be written as

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{\theta}(s') - \pi}{s' - s}\,\mathrm{d}s' = -\frac{\mathrm{i}}{\pi} \int_{-\infty}^{\infty} \frac{\log \hat{q}(s')}{s' - s}\,\mathrm{d}s' = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\log \hat{q}(s') - \mathrm{i}(\hat{\theta}(s') - \pi)}{s' - s}\,\mathrm{d}s',$$

since

$$\int_{-\infty}^{\infty} \frac{\log \hat{q}(s') + \mathrm{i}(\hat{\theta}(s') - \pi)}{s' - s}\,\mathrm{d}s' = 0$$

because

$$\frac{\log \hat{q}(s') - \mathrm{i}(\hat{\theta}(s') - \pi)}{s' - \bar{s}}$$

is analytic in the upper half plane, where $\bar{s}$ is the complex conjugate of s. However, we will use the form (12) for the remainder of the paper.

The normal velocity of the interface is given by

$$v_n = \frac{\sin \hat{\theta}}{1 - \lambda}.$$

Noting that

$$\frac{\partial x}{\partial S} = \cos \hat{\theta}, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}S} = \hat{q}, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}s} = -\frac{1}{\pi s}, \tag{13}$$

where S is arc length, we find on differentiating Equation (6) with respect to arclength that

$$-c\pi \cos \hat{\theta}\, \hat{q} s \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}s} - (1 - \lambda)\hat{q} = \cos \hat{\theta}, \tag{14}$$

with boundary conditions

$$\hat{\theta}(0) = \pi, \qquad \hat{q}(0) = 1/(1 - \lambda), \tag{15}$$

$$\hat{\theta}(1) = \pi/2, \quad \hat{q}(1) = 0. \tag{16}$$

Finally, the dependence on λ may be simplified by setting

$$\theta = \hat{\theta} - \pi, \qquad q = (1-\lambda)\hat{q}, \tag{17}$$

to give

$$2\epsilon \cos\theta\, q s \frac{\mathrm{d}\theta}{\mathrm{d}s} - q = -\cos\theta, \tag{18}$$

$$\log q - \mathrm{i}\theta = -\frac{s}{\pi}\int_0^1 \frac{\theta(s')\,\mathrm{d}s'}{s'(s'-s)}. \tag{19}$$

with boundary conditions

$$\theta(0) = 0, \qquad q(0) = 1, \tag{20}$$

$$\theta(1) = -\pi/2, \quad q(1) = 0, \tag{21}$$

where

$$\epsilon = \frac{c\pi}{2(1-\lambda)} \tag{22}$$

is the dimensionless kinetic undercooling parameter. Once a solution of (18–21) is known, the corresponding value of λ can be deduced from the relation

$$\log(1-\lambda) = \frac{1}{\pi}\int_0^1 \frac{\theta(s')}{s'}\,\mathrm{d}s', \tag{23}$$

obtained from (11) and (15). The Saffman-Taylor solutions are given by

$$q_0 = \sqrt{\frac{1-s}{1+\alpha s}}, \qquad \theta_0 = \cos^{-1}\sqrt{\frac{1-s}{1+\alpha s}}, \tag{24}$$

where

$$\alpha = \frac{2\lambda - 1}{(1-\lambda)^2} \tag{25}$$

is arbitrary.

We could work directly with the variable s in what follows, but this would involve working in a slit plane, since the top and bottom branches of the finger map to the top and bottom of the slit $0 < s < 1$, respectively. We prefer to change variables by setting

$$t = -\sqrt{\frac{1-s}{s}}, \tag{26}$$

giving

$$\frac{\epsilon\cos\theta\, q(1+t^2)}{t}\frac{\mathrm{d}\theta}{\mathrm{d}t} + q = \cos\theta, \tag{27}$$

$$\log q - \mathrm{i}\theta = \frac{2}{\pi}\int_{-\infty}^0 \frac{\bar{t}\theta(\bar{t})}{t^2 - \bar{t}^2}\,\mathrm{d}\bar{t}, \tag{28}$$

$$\theta(-\infty) = 0, \tag{29}$$

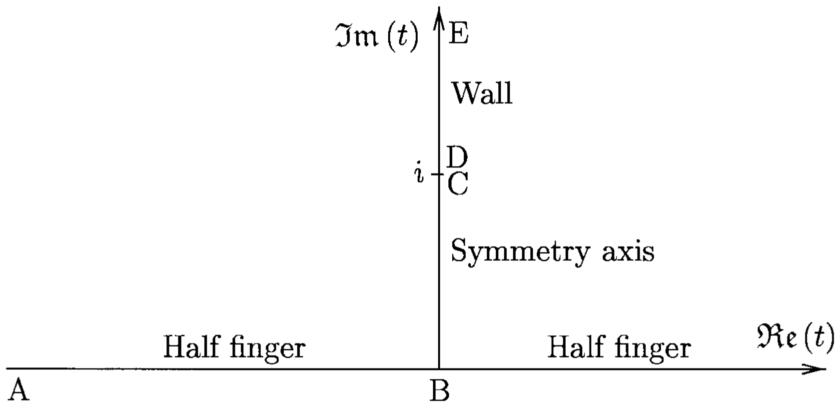


Figure 3. The liquid region in the t plane.

$$\theta(0) = -\pi/2. \tag{30}$$

This maps the top branch of the finger to the negative real axis, the bottom branch to the positive real axis, and the symmetry line to the imaginary axis (see Figure 3). We may now replace the condition at the origin with the condition

$$\theta(\infty) = -\pi \tag{31}$$

if we wish. Note that in (29) and in the related integrals below the integrand is $O(\bar{t}^{-2})$ as $\bar{t} \to -\infty$ so that the integral is well-defined.

3. Analysis as $\epsilon \to 0$ with α of order one

We are interested in the limit $\epsilon \to 0$; in this section we take α to be of order one in (24–25). We begin by expanding in powers of ϵ:

$$\theta \sim \sum_{n=0}^{\infty} \epsilon^n \theta_n, \tag{32}$$

$$q \sim \sum_{n=0}^{\infty} \epsilon^n q_n, \tag{33}$$

giving

$$q_0 = \cos\theta_0, \tag{34}$$

$$\log q_0 - \mathrm{i}\theta_0 = \frac{2}{\pi}\int_{-\infty}^{0} \frac{\bar{t}\theta_0(\bar{t})}{t^2 - \bar{t}^2}\,\mathrm{d}\bar{t}, \tag{35}$$

$$\frac{\cos\theta_0\, q_0(1+t^2)}{t}\frac{\mathrm{d}\theta_0}{\mathrm{d}t} + q_1 = -\theta_1 \sin\theta_0, \tag{36}$$

$$\frac{q_1}{q_0} - \mathrm{i}\theta_1 = \frac{2}{\pi}\int_{-\infty}^{0} \frac{\bar{t}\theta_1(\bar{t})}{t^2 - \bar{t}^2}\,\mathrm{d}\bar{t}, \tag{37}$$

$$\frac{\cos\theta_0\, q_{n-1}(1+t^2)}{t}\frac{\mathrm{d}\theta_0}{\mathrm{d}t} - \frac{\sin\theta_0\,\theta_{n-1}q_0(1+t^2)}{t}\frac{\mathrm{d}\theta_0}{\mathrm{d}t} + \frac{\cos\theta_0\, q_0(1+t^2)}{t}\frac{\mathrm{d}\theta_{n-1}}{\mathrm{d}t} + \cdots$$
$$+ q_n = -\theta_n\sin\theta_0 - \theta_1\theta_{n-1}\cos\theta_0 + \cdots, \qquad n \geq 2, \tag{38}$$

$$\frac{q_n}{q_0} + \cdots - \mathrm{i}\theta_n = \frac{2}{\pi}\int_{-\infty}^{0}\frac{\bar{t}\theta_n(\bar{t})}{t^2-\bar{t}^2}\,\mathrm{d}\bar{t}, \qquad n \geq 2, \tag{39}$$

with boundary conditions

$$\theta_0(-\infty) = 0, \tag{40}$$

$$\theta_0(\infty) = -\pi, \tag{41}$$

$$\theta_n(\pm\infty) = 0, \qquad n \geq 1. \tag{42}$$

The terms on the left-hand side of (38) come from expanding the triple product in $\cos\theta$, q and θ in (27), those on the right-hand side come from expanding $\cos\theta$, and those on the left-hand side of (39) come from expanding $\log q$ in (28). The leading-order solution is the Saffman-Taylor solution (24)

$$q_0 = \cos\theta_0 = -\frac{t}{(1+\alpha+t^2)^{1/2}}, \tag{43}$$

$$\sin\theta_0 = -\frac{(1+\alpha)^{1/2}}{(1+\alpha+t^2)^{1/2}}. \tag{44}$$

Since we are analytically continuing to complex values of t, care is of course required in assigning the branches correctly whenever multivalued functions appear. Here the square roots are real and positive for real t, and this in turn defines the branches in the functions θ_1, q_1 and u below.

We will find that we also require the first correction term (this term is not needed in the corresponding analysis of the surface-tension-regularised problem; its necessity here is due to the fact that the regularisation is first-order rather than second-order). Eliminating q_1 we obtain

$$\theta_1(\mathrm{i}+\tan\theta_0) + \frac{2}{\pi}\int_{-\infty}^{0}\frac{\bar{t}\theta_1(\bar{t})}{t^2-\bar{t}^2}\,\mathrm{d}\bar{t} = -\frac{(1+t^2)}{t}\frac{\mathrm{d}}{\mathrm{d}t}(\sin\theta_0). \tag{45}$$

Substituting in the solution for θ_0 gives

$$\theta_1\left(\mathrm{i}+\frac{(1+\alpha)^{1/2}}{t}\right) + \frac{2}{\pi}\int_{-\infty}^{0}\frac{\bar{t}\theta_1(\bar{t})}{t^2-\bar{t}^2}\,\mathrm{d}\bar{t} = -\frac{(1+t^2)(1+\alpha)^{1/2}}{(1+\alpha+t^2)^{3/2}}, \tag{46}$$

or, for real t,

$$\frac{(1+\alpha)^{1/2}\theta_1}{t} + \frac{2}{\pi}⨍_{-\infty}^{0}\frac{\bar{t}\theta_1(\bar{t})}{t^2-\bar{t}^2}\,\mathrm{d}\bar{t} = -\frac{(1+t^2)(1+\alpha)^{1/2}}{(1+\alpha+t^2)^{3/2}}. \tag{47}$$

This equation can be solved explicitly by the Wiener-Hopf technique (we omit the details; see for example [19, Chapter 8] for a description of the technique), to give

$$\theta_1 = -\frac{(1+\alpha)^{1/2}t}{\pi(1+\alpha+t^2)}\left[C + \frac{\pi(1+\alpha)^{1/2}(1+t^2)}{(1+\alpha+t^2)^{3/2}} + \frac{2(1+t^2)}{(1+\alpha+t^2)} + \frac{t(1+t^2)}{(1+\alpha+t^2)^{3/2}}\log\left(\frac{1+\alpha+2t^2-2t(1+\alpha+t^2)^{1/2}}{1+\alpha}\right)\right], \tag{48}$$

$$q_1 = \frac{(1+\alpha)^{1/2}\theta_1}{(1+\alpha+t^2)^{1/2}} + \frac{t(1+t^2)(1+\alpha)^{1/2}}{(1+\alpha+t^2)^2}, \tag{49}$$

where the constant C is arbitrary at this point. In fact, C corresponds to a perturbation to α and is therefore related to the first-order correction to the eigenvalues α_n, which here we will determine only to leading order in ϵ.

In order to truncate the expansions (32), (33) optimally we need to know the behaviour of θ_n and q_n as $n \to \infty$. From Equation (38) we see that the set of singular points of θ_n for all n will be those of θ_0, together with the points $t = \pm\mathrm{i}$. However, since each time we calculate a new term in the asymptotic expansion we differentiate the previous term, if θ_{n-1} has a singularity of strength p then θ_n will have a singularity of strength $p+1$. Thus we expect there to be factorial/power divergence in the large n behaviour of θ_n and q_n. Following [17] we therefore seek a solution as $n \to \infty$ of the form

$$\theta_n \sim \Theta\frac{\Gamma(n+\gamma)}{u^{n+\gamma}}, \tag{50}$$

where u, γ and Θ are possibly functions of t but are independent of n.

Now, in (39) we need to estimate the relative sizes of θ_n and

$$I = \int_{-\infty}^{0}\frac{\bar{t}\theta_n(\bar{t})}{t^2-\bar{t}^2}\,\mathrm{d}\bar{t}. \tag{51}$$

Substituting (50) in (51) we have

$$I \sim \Theta\Gamma(n+\gamma)\int_{-\infty}^{0}\frac{\Theta\bar{t}}{(t^2-\bar{t}^2)u^{\gamma}}\mathrm{e}^{-n\log u}\,\mathrm{d}\bar{t}, \qquad \text{as } n \to \infty. \tag{52}$$

We will verify a posteriori that the dominant contribution to the integral comes from the upper end point. Then I will be exponentially subdominant to θ_n wherever $u(t)$ is smaller that $u(0)$. This will certainly be true near the singularities in the complex plane (where $u = 0$), and in fact it will hold throughout the region in which we are interested. Thus we have

$$\frac{q_n}{q_0} - \frac{q_1 q_{n-1}}{q_0^2} \sim \mathrm{i}\theta_n \text{ as } n \to \infty. \tag{53}$$

Hence

$$q_n \sim \mathrm{i}q_0\theta_n + \mathrm{i}q_1\theta_{n-1} \text{ as } n \to \infty. \tag{54}$$

Substituting (50) and (54) in (38) we find that at leading order as $n \to \infty$

$$u' = \frac{t(\mathrm{i}q_0 + \sin\theta_0)}{q_0\cos\theta_0(1+t^2)}, \tag{55}$$

giving

$$u = -\int_{\mathrm{i}(1+\alpha)^{1/2}}^{t} \frac{\left((1+\alpha)^{1/2}+\mathrm{i}\bar{t}\right)^{3/2}\left((1+\alpha)^{1/2}-\mathrm{i}\bar{t}\right)^{1/2}}{\bar{t}(1+\bar{t}^2)}\,\mathrm{d}\bar{t}. \tag{56}$$

The integral can be evaluated explicitly, but the resulting expression is rather cumbersome; it is given in Appendix 3 for completeness.

To evaluate γ and Θ we need to go to the next order in n. The easiest way to proceed is first to use u as the independent variable [17]. Equations (18) and (19) become

$$\epsilon\frac{\mathrm{d}\theta}{\mathrm{d}u} = \frac{t(u)}{u'(1+t(u)^2)}\left(\frac{1}{q}-\frac{1}{\cos\theta}\right), \tag{57}$$

$$\log q - \mathrm{i}\theta = \frac{2}{\pi}\int_{-\infty}^{0}\frac{\bar{t}\theta(\bar{t})}{t(u)^2-\bar{t}^2}\,\mathrm{d}\bar{t}, \tag{58}$$

while (38) becomes

$$\frac{\mathrm{d}\theta_{n-1}}{\mathrm{d}u} = \frac{t}{u'(1+t^2)}\left(-\frac{q_n}{q_0^2}+\frac{2q_{n-1}q_1}{q_0^3}-\theta_n\frac{\sin\theta_0}{\cos^2\theta_0}-\theta_1\theta_{n-1}\frac{1+2\tan^2\theta_0}{\cos\theta_0}\right)+\cdots; \tag{59}$$

in (57) and (59) u' denotes $u'(t(u))$. Simplifying, we find that

$$\frac{\mathrm{d}\theta_{n-1}}{\mathrm{d}u}+\theta_n = \frac{t\theta_{n-1}}{u'(1+t^2)\cos^2\theta_0}\left(\mathrm{i}q_1-(\cos\theta_0+2\sin\theta_0\tan\theta_0)\theta_1\right)+\cdots. \tag{60}$$

Substituting in the *ansatz* (50), we find that the terms of order $n\log n$ give $\gamma'=0$, so that γ is constant. Equating the terms of order n we find

$$\frac{\mathrm{d}\Theta}{\mathrm{d}u} = \frac{\Theta}{u'}\frac{t}{(1+t^2)\cos^2\theta_0}\left(\mathrm{i}q_1-(\cos\theta_0+2\sin\theta_0\tan\theta_0)\theta_1\right). \tag{61}$$

Hence

$$\log\Theta = \int\frac{t}{(1+t^2)\cos^2\theta_0}\left(\mathrm{i}q_1-(\cos\theta_0+2\sin\theta_0\tan\theta_0)\theta_1\right)\,\mathrm{d}t. \tag{62}$$

Thus

$$\theta_n \sim \frac{\Lambda\Gamma(n+\gamma)}{u^{n+\gamma}}\mathrm{e}^{F(t)}, \tag{63}$$

where

$$F(t) = \int_{\mathrm{i}}^{t}\frac{t}{(1+t^2)\cos^2\theta_0}\left(\mathrm{i}q_1-(\cos\theta_0+2\sin\theta_0\tan\theta_0)\theta_1\right)\,\mathrm{d}t. \tag{64}$$

The choice of starting point for the integral is arbitrary: changing it simply changes the value of constant Λ.

Now from (56) and (43), (44), as $t\to\mathrm{i}(1+\alpha)^{1/2}$ we have

$$u \sim \frac{2^{3/2}\mathrm{e}^{\mathrm{i}\pi/4}}{5\alpha(1+\alpha)^{1/4}}(t-\mathrm{i}(1+\alpha)^{1/2})^{5/2}, \tag{65}$$

$$\theta_0 \sim (\mathrm{i}/2)\log(t-\mathrm{i}(1+\alpha)^{1/2}). \tag{66}$$

Using (48–49) in (64) we may calculate the behaviour of $\mathrm{e}^{F(t)}$ as $t\to\mathrm{i}(1+\alpha)^{1/2}$. The analysis is relegated to Appendix 3, but the result is

$$e^{F(t)} \sim B e^{i\pi/4}(t - \mathrm{i}(1+\alpha)^{1/2})^{1/2},$$

as $t \to \mathrm{i}(1+\alpha)^{1/2}$, for some real constant B. Since θ_0 is logarithmic in u as $u \to 0$ Equation (63) implies

$$\gamma = \tfrac{1}{5}. \tag{67}$$

Having found the asymptotic behaviour of θ_n as $n \to \infty$, the next step is to truncate the asymptotic series and study the behaviour of the remainder. Truncating after N terms we have

$$\theta = \sum_{n=0}^{N-1} \epsilon^n \theta_n + R_N, \tag{68}$$

where

$$\epsilon \frac{\mathrm{d}R_N}{\mathrm{d}u} + R_N + \epsilon R_N \frac{t}{u'(1+t^2)\cos^2\theta_0}\left((\cos\theta_0 + 2\sin\theta_0\tan\theta_0)\theta_1 - \mathrm{i}q_1\right) + \cdots \sim \epsilon^N \theta_N, \tag{69}$$

as $N \to \infty$, $\epsilon \to 0$. The homogeneous version of this equation has a solution with the following asymptotic behaviour as $\epsilon \to 0$:

$$R_N \sim e^{-u/\epsilon} e^{F(t)}.$$

We aim to show that a multiple of this is switched on across the Stokes line where u is real and positive. We define the Stokes multiplier A by setting

$$R_N = e^{-u/\epsilon} e^{F(t)} A, \tag{70}$$

which in (69) gives

$$\epsilon e^{-u/\epsilon} \frac{\mathrm{d}A}{\mathrm{d}u} + \cdots \sim \frac{\epsilon^N \Lambda \Gamma(N+\gamma)}{u^{N+\gamma}}. \tag{71}$$

Following [17] we let $u = r e^{i\vartheta}$, and write

$$\frac{\mathrm{d}}{\mathrm{d}u} = -\frac{\mathrm{i}e^{-i\vartheta}}{r}\frac{\mathrm{d}}{\mathrm{d}\vartheta}.$$

The right-hand side of (71) is minimal for $N \sim r/\epsilon$. We therefore set $N = r/\epsilon + \mu$, where $\mu \in [0, 1)$. Then, as $\epsilon \to 0$, (71) becomes

$$\frac{\mathrm{d}A}{\mathrm{d}\vartheta} \sim \frac{\mathrm{i}\sqrt{2\pi}\, e^{re^{\mathrm{i}\vartheta}/\epsilon} \Lambda e^{-r/\epsilon} r^{1/2}}{\epsilon^{\gamma+1/2} e^{\mathrm{i}\vartheta r/\epsilon} e^{\mathrm{i}\vartheta(\mu+\gamma-1)}}. \tag{72}$$

The right-hand side is exponentially small except on the Stokes line $\vartheta = 2\pi k$, $k \in \mathbb{Z}$, where it is algebraic in ϵ. Hence A will undergo a rapid transition near $\vartheta = 2\pi k$. To examine this transition we introduce the local scaling $\vartheta = 2\pi k + \delta\bar{\vartheta}$, giving

$$\frac{\mathrm{d}A}{\mathrm{d}\bar{\vartheta}} \sim \frac{\mathrm{i}\sqrt{2\pi}\, \delta r^{1/2} \Lambda e^{-r\delta^2\bar{\vartheta}^2/(2\epsilon)}}{\epsilon^{\gamma+1/2} e^{2\pi \mathrm{i}k\gamma}}. \tag{73}$$

We see that the correct scalings are $\vartheta = \epsilon^{1/2}\bar{\vartheta}$, $A = \hat{A}/\epsilon^{\gamma}$, giving

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}\bar{\vartheta}} \sim \frac{\mathrm{i}\sqrt{2\pi}\, r^{1/2} \Lambda e^{-r\bar{\vartheta}^2/2}}{e^{2\pi \mathrm{i}k\gamma}}. \tag{74}$$

Thus

$$\hat{A} = a + \frac{\mathrm{i}\sqrt{2\pi}\,\Lambda}{\mathrm{e}^{2\pi \mathrm{i}k\gamma}} \int_{-\infty}^{\vartheta\sqrt{r}} \mathrm{e}^{-x^2/2}\,\mathrm{d}x, \tag{75}$$

that is, we have the familiar error function smoothing [20]. Matching with the outer solution away from the Stokes line there is a jump in A given by

$$A(\vartheta = 2\pi k+) - A(\vartheta = 2\pi k-) = \frac{2\pi\,\mathrm{i}\Lambda \mathrm{e}^{-2\pi \mathrm{i}k\gamma}}{\epsilon^{\gamma}}. \tag{76}$$

Let us now examine the implications of these Stokes lines for the problem (27–30). The Stokes lines are given by u real and positive.

We consider first the case $\alpha < 0$, corresponding to $\lambda < 1/2$. Figure 4(a) shows typical Stokes lines in this case. There is a Stokes line emerging downwards from $t = \mathrm{i}\sqrt{1+\alpha}$ and another emerging upwards from $t = -\mathrm{i}\sqrt{1+\alpha}$, both parallel to the imaginary axis.

On the Stokes line emerging from $t = \mathrm{i}\sqrt{1+\alpha}$ the argument of u is 0 and across it

$$\frac{2\pi\,\mathrm{i}\Lambda \mathrm{e}^{-u(t)/\epsilon}}{\epsilon^{\gamma}} \mathrm{e}^{F(t)} \tag{77}$$

is switched on. Similarly, or by symmetry, as we cross the Stokes line emerging from $t = -\mathrm{i}\sqrt{1+\alpha}$

$$-\frac{2\pi\,\mathrm{i}\bar{\Lambda} \mathrm{e}^{-\bar{u}(t)/\epsilon}}{\epsilon^{\gamma}} \mathrm{e}^{\bar{F}(t)} \tag{78}$$

is switched on. On the real axis where these Stokes lines meet

$$\mathfrak{Re}\left(\frac{2\pi\,\mathrm{i}\Lambda \mathrm{e}^{-u(t)/\epsilon}}{\epsilon^{\gamma}} \mathrm{e}^{F(t)}\right) \tag{79}$$

is switched on. Since this does not decay sufficiently rapidly at infinity there are no solutions with $\lambda < 1/2$.

Let us now consider the case $\alpha > 0$, corresponding to $\lambda > 1/2$. In this case typical Stokes lines are shown in Figure 4(b).

This time there are two Stokes lines emerging downwards from $t = \mathrm{i}\sqrt{1+\alpha}$, one either side of the imaginary axis, and two similar Stokes lines emerging from $t = -\mathrm{i}\sqrt{1+\alpha}$.

On the left-hand Stokes lines the argument of u is -2π, and as we cross their intersection point on the real axis

$$\mathfrak{Re}\left(\frac{2\pi\,\mathrm{i}\Lambda \mathrm{e}^{-u/\epsilon}}{\mathrm{e}^{-2\pi \mathrm{i}\gamma}\epsilon^{\gamma}} \mathrm{e}^{F(t)}\right) \tag{80}$$

is switched on. On the right-hand Stokes lines the argument of u is zero, and as we cross the point at which they meet the real axis,

$$\mathfrak{Re}\left(\frac{2\pi\,\mathrm{i}\Lambda \mathrm{e}^{-u/\epsilon}}{\epsilon^{\gamma}} \mathrm{e}^{F(t)}\right) \tag{81}$$

is switched on. The condition for a solution is that these two functions exactly cancel as $t \to \infty$. However, now we must remember that the branch of the function u in (80) is not the same as that in (81), due to the fact that we have circled the branch point at $t = \mathrm{i}$. In fact, if we represent the first branch of u by u_1 and the second by u_2, then

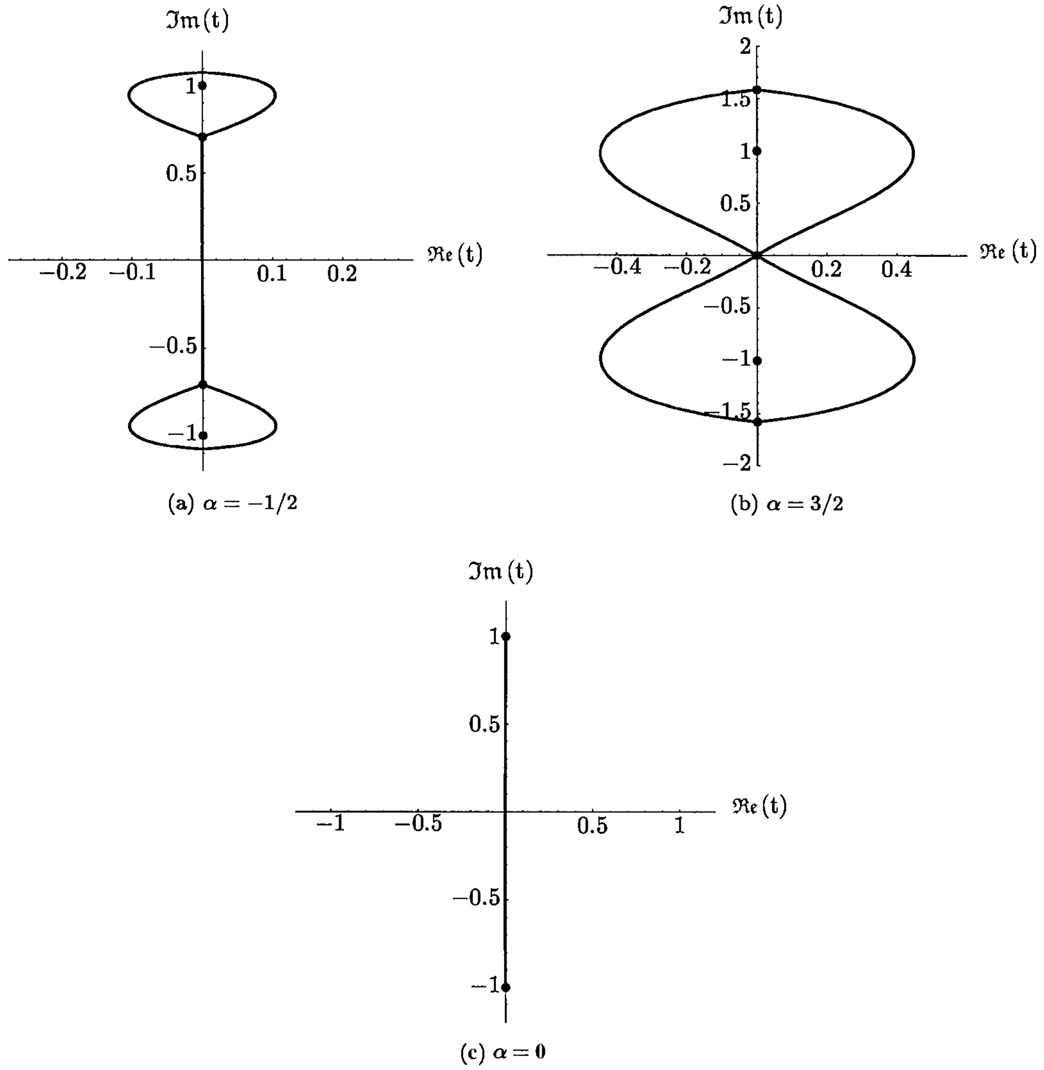


Figure 4. The Stokes lines, which are given by u real and positive, in the complex t plane for different values of α.

$$u_1 - u_2 = \pi \mathrm{i} \alpha^{1/2} \left((1+\alpha)^{1/2} - 1 \right), \tag{82}$$

since, from (56), the difference is exactly the residue due to the pole at $t = \mathrm{i}$. Hence, the asymptotic condition for a solution to exist is that

$$\frac{u_1}{\mathrm{i}\epsilon} - 2\pi \mathrm{i}\gamma - \frac{u_2}{\mathrm{i}\epsilon} = (2k+1)\pi \mathrm{i}, \tag{83}$$

i.e.,

$$\alpha^{1/2} \left((1+\alpha)^{1/2} - 1 \right) = \epsilon(2k + 1 + 2\gamma) = \epsilon(2k + 7/5), \tag{84}$$

where $k \in \mathbb{Z}$, remembering that $\gamma = 1/5$.

Alternatively, we may write the condition for there to be a solution as $\theta(0) = -\pi/2$. However, from (56) we see that in fact $\Re\mathfrak{e}(u) \to \infty$ as $t \to 0$, so that the exponentially small correction term in fact vanishes at the origin for any α. The behaviour of the Stokes lines in the vicinity of the origin is unusual, and an inner analysis is needed in order to determine the solvability condition by imposing $\theta(0) = -\pi/2$. In Appendix 2 we examine a paradigm problem which elucidates this behaviour. Naively we might expect the solvability condition to be

$$\cos\left(\Phi + \pi/2 - \frac{\mathfrak{Im}(u_1(0))}{\epsilon} + 2\pi\gamma\right) = 0 = \cos\left(\Phi + \pi/2 - \frac{\mathfrak{Im}(u_2(0))}{\epsilon}\right), \tag{85}$$

where $\Lambda = R\mathrm{e}^{i\Phi}$. Since

$$\mathfrak{Im}(u_1(0)) = \frac{\pi\alpha^{1/2}}{2}\left((1+\alpha)^{1/2} - 1\right), \tag{86}$$

this gives

$$\alpha^{1/2}\left((1+\alpha)^{1/2} - 1\right) = \epsilon\left(2k + \frac{2\Phi}{\pi} + 4\gamma\right) = \epsilon\left(2k + \frac{2\Phi}{\pi} + 4/5\right), \tag{87}$$

where $k \in \mathbb{Z}$. For this to be consistent with (84) we must have $\Phi = (2m + 1 - 2\gamma)\pi/2 = m\pi + 3\pi/10$. We will see in the next section that this is indeed the case, by matching with an inner solution in the vicinity of the singularity at $t = \mathrm{i}\sqrt{1+\alpha}$. We emphasize though that we have been able to derive the condition (84) for a solution to exist without solving either the inner problem in the vicinity of the singularity, or the inner problem in the vicinity of the origin.

Thus we see that for any fixed k, *i.e.*, for any particular solution branch, $\alpha \to 0$ as $\epsilon \to 0$, corresponding to $\lambda \to 1/2$. However, for each fixed ϵ there are infinitely many solution branches, which cluster around $\alpha = \infty$, corresponding to $\lambda = 1$. The first ten solution branches in the corresponding asymptotic regime are shown in Figure 5.

For $\alpha \to 0$ we find

$$\alpha^{3/2} \sim 2\epsilon(2k + 7/5). \tag{88}$$

Now, for the lower solution branches, (and in particular for the lowest solution branch $k = 0$, which is the one we expect to be observable in practice) we have $\alpha = O(\epsilon^{2/3})$, while our analysis has been performed under the assumption that $\alpha = O(1)$ as $\epsilon \to 0$. In Section 5 we will consider the case $\alpha = O(\epsilon^{2/3})$.

4. The constant Λ

To complete the analysis of the case $\alpha = O(1)$ we need to evaluate Λ. To do this we may write down the full Laurent expansion for θ_n and compare this for large n with our asymptotic behaviour (50). Since we require just one number, we need to compare for just one value of s. We therefore choose the simplest value of s to make this comparison, namely at the singularity $s = \mathrm{i}\sqrt{1+\alpha}$. To evaluate the Laurent expansion of θ as $s \to \mathrm{i}\sqrt{1+\alpha}$ we take advantage of matched asymptotics. In that language we require one term in the inner limit of each order of the outer expansion. It is therefore enough for us to have the outer limit of the one term inner expansion. The correct scaling for the inner problem is

$$t = \mathrm{i}(1+\alpha)^{1/2} + \epsilon^{2/5}z, \tag{89}$$

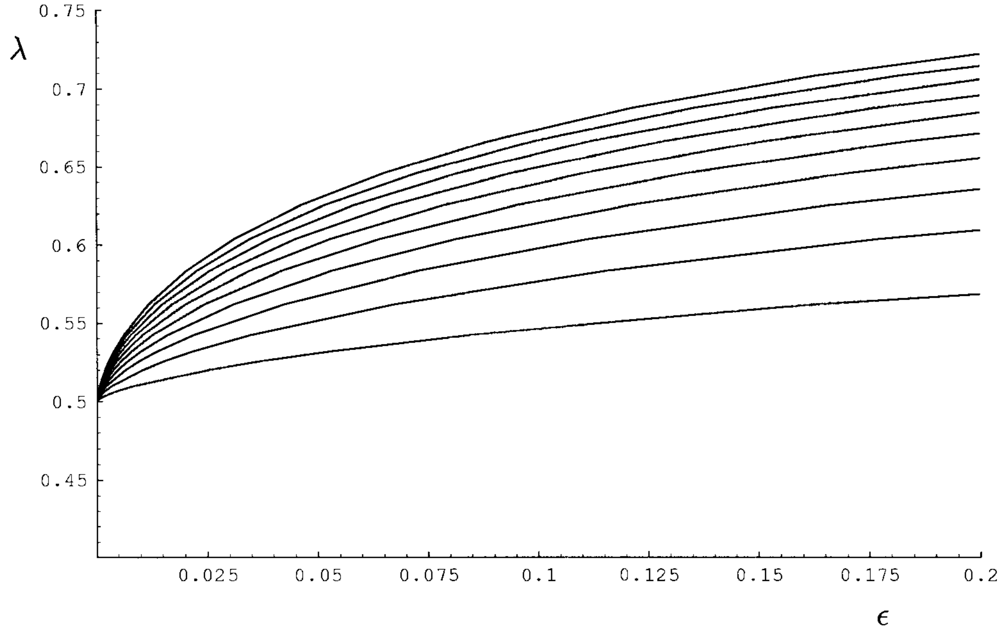


Figure 5. The relative finger width λ as a function of kinetic undercooling strength ϵ for the first ten solution branches. The curves are valid asymptotically in the limit $\epsilon \to 0$, ϵk order one. Of course, this implies that the lower branches in the figure do not provide accurate representations of the asymptotic behaviour; the analysis of Sections 5 and 6 concerns the case $\epsilon \to 0$ with $n = O(1)$ and thus encompasses in particular the small ϵ behaviour of the most important (*i.e.*, stable) branch, namely $n = 1$.

giving

$$q_0 = \cos\theta_0 \sim \frac{\mathrm{e}^{-3\mathrm{i}\pi/4}(1+\alpha)^{1/4}}{2^{1/2}z^{1/2}\epsilon^{1/5}}, \tag{90}$$

$$\sin\theta_0 \sim -\frac{\mathrm{e}^{-\mathrm{i}\pi/4}(1+\alpha)^{1/4}}{2^{1/2}z^{1/2}\epsilon^{1/5}}, \tag{91}$$

$$\theta_0 \sim \frac{\mathrm{i}}{2}\log z - \mathrm{i}\log\left(\frac{2^{1/2}(1+\alpha)^{1/4}\mathrm{e}^{-3\mathrm{i}\pi/4}}{\epsilon^{1/5}}\right). \tag{92}$$

Writing (28) in inner coordinates gives

$$\log q - \mathrm{i}\theta = \frac{2}{\pi}\int_{-\infty}^{0}\frac{\bar{t}\theta(\bar{t})}{t^2-\bar{t}^2}\,\mathrm{d}\bar{t}, \tag{93}$$

$$\sim -\log 2 - \frac{\mathrm{i}\epsilon^{2/5}z}{2(1+\alpha)^{1/2}}. \tag{94}$$

Therefore, we define the inner variable ϕ by

$$q = \frac{\mathrm{e}^{-3\mathrm{i}\pi/4}(1+\alpha)^{1/4}}{2^{1/2}\epsilon^{1/5}z^{1/2}}\phi, \tag{95}$$

giving, to leading order in the inner region

$$-\frac{\alpha \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2}\,(1+\alpha)^{1/4}}{z}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi_0}{z^{1/2}}\right) = 1 - \frac{1}{\phi_0^2}. \tag{96}$$

The rescaling

$$z = \left(\alpha \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2}\,(1+\alpha)^{1/4}\right)^{2/5}\zeta, \tag{97}$$

then gives

$$-\frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\phi_0}{\zeta^{1/2}}\right) = 1 - \frac{1}{\phi_0^2}. \tag{98}$$

We require the outer limit, that is, we need to expand ϕ as $\zeta \to \infty$. We find

$$\phi_0 \sim \sum_{n=0}^{\infty} \frac{A_n}{\zeta^{5n/2}} = \sum_{n=0}^{\infty} \frac{A_n}{z^{5n/2}}\left(\alpha \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2}\,(1+\alpha)^{1/4}\right)^{n}, \tag{99}$$

where the A_n satisfy the recurrence relation

$$\sum_{m=1}^{n}\left(\frac{5m}{2} - 2\right) A_{m-1} \sum_{k=0}^{n-m} A_{n-m-k} A_k = \sum_{m=0}^{n} A_{n-m} A_m, \quad n \geq 1, \tag{100}$$

with

$$A_0 = 1. \tag{101}$$

Returning to the outer expansion, since $q_n \sim \mathrm{i}\theta_n q_0$ as $n \to \infty$, we have

$$\epsilon^n q_n \sim \frac{\mathrm{i}\epsilon^n \Lambda q_0 \Gamma(n+\gamma)}{u^{n+\gamma}} e^{F(t)} \quad \text{as } n \to \infty \tag{102}$$

$$\sim \frac{\mathrm{i}\Lambda \mathrm{e}^{-3\mathrm{i}\pi/4}(1+\alpha)^{1/4}}{2^{1/2} z^{1/2}\epsilon^{1/5}} B \mathrm{e}^{\mathrm{i}\pi/4}\epsilon^{1/5} z^{1/2}\left(\frac{5\alpha(1+\alpha)^{1/4}}{2^{3/2}\mathrm{e}^{\mathrm{i}\pi/4}\epsilon z^{5/2}}\right)^{\gamma}\Gamma(n+\gamma)\left(\frac{5\alpha(1+\alpha)^{1/4}}{2^{3/2}\mathrm{e}^{\mathrm{i}\pi/4}\epsilon z^{5/2}}\right)^{n}, \tag{103}$$

as $\epsilon \to 0$ with $z = O(1)$. Comparing (95), (99) and (103) we see

$$\Lambda = -\frac{\mathrm{i}}{B\mathrm{e}^{\mathrm{i}\pi/4}}\left(\frac{2^{3/2}\mathrm{e}^{\mathrm{i}\pi/4}}{5\alpha(1+\alpha)^{1/4}}\right)^{\gamma}\lim_{n\to\infty}\left(\frac{4}{5}\right)^{n}\frac{A_n}{\Gamma(n+\gamma)}. \tag{104}$$

From (100) we find that A_n is real and positive as $n \to \infty$. Hence

$$\Phi = -\frac{3\pi}{4} + \frac{\gamma\pi}{4} = -7\pi/10, \tag{105}$$

which confirms the corresponding result of Section 3.

5. Analysis as $\epsilon \to 0$ with α order $\epsilon^{2/3}$

5.1. Algebraic expansion

In the Section 3 we gave evidence that for n of order one we have $\alpha = O(\epsilon^{2/3})$ as $\epsilon \to 0$. That analysis assumed that α was order one, so that the singularities at $t = \mathrm{i}$ and $t = \mathrm{i}\sqrt{1+\alpha}$ were well separated in the outer region. In this section we consider the case $\alpha = O(\epsilon^{2/3})$, which requires a separate analysis, via a similar Stokes line approach. We set

$$\alpha = \epsilon^{2/3} a. \tag{106}$$

The expansion of θ and q now proceeds in powers of $\epsilon^{1/3}$:

$$q \sim \sum_{n=0}^{\infty} \epsilon^{n/3} q_n, \qquad \theta \sim \sum_{n=0}^{\infty} \epsilon^{n/3} \theta_n. \tag{107}$$

The first three terms in this expansion may be obtained by expanding (43), (44) in powers of ϵ after substituting in (106), since the coupling between terms does not appear until $O(\epsilon)$:

$$q_0 = \cos\theta_0 = \sqrt{1-s} = -\frac{t}{(1+t^2)^{1/2}}, \tag{108}$$

$$q_1 = 0, \tag{109}$$

$$\sin\theta_0 = -\sqrt{s} = -\frac{1}{(1+t^2)^{1/2}}, \tag{110}$$

$$\theta_1 = 0, \tag{111}$$

$$\theta_2 = \frac{at}{2(1+t^2)}. \tag{112}$$

We will also need the solution for θ_3, which is given by the $\alpha \to 0$ limit of (48), namely

$$\theta_3 = -\frac{Ct}{\pi(1+t^2)} - \frac{t}{(1+t^2)^{3/2}} - \frac{t^2}{\pi(1+t^2)^{3/2}} \log(t - (1+t^2)^{1/2})^2, \tag{113}$$

where C is arbitrary (as before, C can be related to the first order correction to a). Writing

$$\epsilon \frac{(1+t^2)}{t}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{1}{\cos\theta} = \frac{1}{q}, \qquad \log q - \mathrm{i}\theta = \frac{2}{\pi}\int_{-\infty}^{0} \frac{t'\theta(t')\,\mathrm{d}t'}{t^2-(t')^2},$$

we find that the equations for θ_n, q_n are

$$\begin{aligned} \frac{(1+t^2)}{t}\frac{\mathrm{d}\theta_{n-3}}{\mathrm{d}t} + \frac{\sin\theta_0}{\cos^2\theta_0}\theta_n + \left(\frac{1+\sin^2\theta_0}{\cos^3\theta_0}\right)(\theta_2\theta_{n-2} + \theta_3\theta_{n-3}) + \cdots = \\ -\frac{q_n}{q_0^2} + \frac{2q_2q_{n-2}}{q_0^3} + \frac{2q_3q_{n-3}}{q_0^3} + \cdots, \end{aligned} \tag{114}$$

$$\frac{q_n}{q_0} - \frac{q_2q_{n-2}}{q_0^2} - \frac{q_3q_{n-3}}{q_0^2} + \cdots = \mathrm{i}\theta_n. \tag{115}$$

From (115) we find

$$q_n \sim \mathrm{i}q_0\theta_n + \mathrm{i}q_2\theta_{n-2} + \mathrm{i}q_3\theta_{n-3} + \cdots. \tag{116}$$

Using (116) and the relations

$$q_0 = \cos\theta_0, \quad q_2 = -\sin\theta_0\theta_2, \quad q_3 = -\sin\theta_0\theta_3 - \cos^2\theta_0 \frac{(1+t^2)}{t}\frac{\mathrm{d}\theta_0}{\mathrm{d}t},$$

we have

$$\frac{\mathrm{d}\theta_{n-3}}{\mathrm{d}t} + \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}}{\cos^2\theta_0}\frac{t\theta_n}{1+t^2} + \left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\theta_2\theta_{n-2}}{1+t^2}$$
$$+ \left(\left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\theta_3}{1+t^2} + \mathrm{i}\frac{\mathrm{d}\theta_0}{\mathrm{d}t}\right)\theta_{n-3} + \cdots = 0. \tag{117}$$

As $n \to \infty$, θ_n has the behaviour

$$\theta_n \sim \frac{\Theta n^{n/3+\gamma}\mathrm{e}^{-n/3+\beta n^{1/3}}}{u^{n/3}}, \tag{118}$$

where Θ, u, β and γ may be functions of t but are independent of n. The complicated asymptotic behaviour (118) of θ_n (by comparison to the usual factorial/power) is a direct consequence of the fact that kinetic undercooling is a first-order rather than second-order regurisation. In fact we will find when matching with the inner solution in the vicinity of $t = \mathrm{i}$ that we must be careful to allow all three branches of $u^{n/3}$ in the denominator, so that we write instead

$$\theta_n \sim \frac{\Theta_k n^{n/3+\gamma}\mathrm{e}^{-n/3+\beta n^{1/3}}}{\mathrm{e}^{2\pi\mathrm{i}nk/3}u^{n/3}}, \qquad k = 0, 1, 2. \tag{119}$$

Substituting (119) in (117) and dividing by θ_n gives

$$\left(1-\frac{3}{n}\right)^{n/3+\gamma}\frac{\mathrm{e}}{n-3}\mathrm{e}^{\beta\left((n-3)^{1/3}-n^{1/3}\right)}\left[\frac{u}{\Theta_k}\frac{\mathrm{d}\Theta_k}{\mathrm{d}t} - \left(\frac{n}{3}-1\right)\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\mathrm{d}\gamma}{\mathrm{d}t}\log(n-3)u + \frac{\mathrm{d}\beta}{\mathrm{d}t}(n-3)^{1/3}u\right]$$
$$+\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}t}{\cos^2\theta_0(1+t^2)} + \left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\theta_2}{1+t^2}\frac{\mathrm{e}^{4\pi\mathrm{i}k/3}u^{2/3}\mathrm{e}^{2/3}}{(n-2)^{2/3}}\left(1-\frac{2}{n}\right)^{n/3+\gamma}\mathrm{e}^{\beta\left((n-2)^{1/3}-n^{1/3}\right)}$$
$$+\left(\left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\theta_3}{1+t^2} + \mathrm{i}\frac{\mathrm{d}\theta_0}{\mathrm{d}t}\right)\frac{u\mathrm{e}}{n-3}\left(1-\frac{3}{n}\right)^{n/3+\gamma}\mathrm{e}^{\beta\left((n-3)^{1/3}-n^{1/3}\right)} + \cdots = 0.$$

Noting that

$$\begin{aligned}
\left(1-\frac{3}{n}\right)^{n/3+\gamma} &= \mathrm{e}^{-1}\left(1-\left(3\gamma+\tfrac{3}{2}\right)\tfrac{1}{n}+\cdots\right),\\
\left(1-\frac{2}{n}\right)^{n/3+\gamma} &= \mathrm{e}^{-2/3}+\cdots,\\
\mathrm{e}^{\beta\left((n-3)^{1/3}-n^{1/3}\right)} &= 1-\beta n^{-2/3}+\cdots
\end{aligned}$$

we find that the $O(n^0)$ balance gives

$$\frac{1}{3}\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{i}\mathrm{e}^{-i\theta_0}t}{\cos^2\theta_0(1+t^2)} = -\frac{(1+\mathrm{i}t)^{1/2}}{t(1-\mathrm{i}t)^{1/2}}.$$

Hence

$$u = -3\int_{\mathrm{i}}^{t}\frac{(1+\mathrm{i}t)^{1/2}}{t(1-\mathrm{i}t)^{1/2}}\,\mathrm{d}t. \tag{120}$$

We may evaluate the integral to give

$$u = \frac{3(\mathrm{i}-1)\pi}{2} + 3\log\left(1+(1+t^2)^{1/2}\right) - 3\log t - 3\mathrm{i}\log\left(t+(1+t^2)^{1/2}\right). \tag{121}$$

Equating coefficients at $O(n^{-2/3})$ gives

$$\frac{\beta}{3}\frac{\mathrm{d}u}{\mathrm{d}t} + u\frac{\mathrm{d}\beta}{\mathrm{d}t} + \left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\mathrm{e}^{4\pi\mathrm{i}k/3}u^{2/3}\theta_2}{1+t^2} = 0.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{2\pi\mathrm{i}k/3}u^{1/3}\beta) = -\left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\theta_2}{1+t^2} = \frac{a(2-\mathrm{i}t)}{2t(1-\mathrm{i}t)^{3/2}(1+\mathrm{i}t)^{1/2}}.$$

Hence

$$\beta = \frac{\mathrm{e}^{-2\pi\mathrm{i}k/3}}{u^{1/3}}\int_{\mathrm{i}}^{t}\frac{a(2-\mathrm{i}t)\,\mathrm{d}t}{2t(1-\mathrm{i}t)^{3/2}(1+\mathrm{i}t)^{1/2}}. \tag{122}$$

Equating coefficients at $O(n^{-1}\log n)$ we find $\mathrm{d}\gamma/\mathrm{d}t = 0$, so that γ is in fact constant. Finally, equating coefficients at $O(n^{-1})$ we find

$$\frac{u}{\Theta_k}\frac{\mathrm{d}\Theta_k}{\mathrm{d}t} + \left(\gamma+\frac{1}{2}\right)\frac{\mathrm{d}u}{\mathrm{d}t} + \left(\left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\theta_3}{1+t^2} + \mathrm{i}\frac{\mathrm{d}\theta_0}{\mathrm{d}t}\right)u = 0.$$

Hence

$$\Theta_k = \frac{\mathrm{e}^{-\mathrm{i}\theta_0}}{u^{\gamma+1/2}}\exp\left(-\int^t\left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\theta_3}{1+t^2}\,\mathrm{d}t\right) = \frac{G_k}{u^{\gamma+1/2}}, \tag{123}$$

say. Now, as $t\to\mathrm{i}$,

$$\begin{aligned}
\theta_3 &\sim \frac{\mathrm{e}^{3\mathrm{i}\pi/4}}{2^{1/2}(t-\mathrm{i})^{3/2}},\\
-\int^t\left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\theta_3}{1+t^2}\,\mathrm{d}t &\sim \frac{3}{2(t-\mathrm{i})},\\
\mathrm{e}^{-\mathrm{i}\theta_0} &\sim \frac{\mathrm{e}^{3\mathrm{i}\pi/4}(t-\mathrm{i})^{1/2}}{2^{1/2}},
\end{aligned}$$

so that

$$\Theta_k u^{\gamma+1/2} \sim \text{const.}\times(t-\mathrm{i})^2.$$

Since $u\sim 2^{1/2}\mathrm{e}^{3\mathrm{i}\pi/4}(t-\mathrm{i})^{3/2}$ and $\theta_0\sim(\mathrm{i}/2)\log(t-\mathrm{i})$ as $t\to\mathrm{i}$ we therefore have that

$$\frac{3}{2}\left(\gamma+\frac{1}{2}\right) = 2, \qquad \textit{i.e.}, \quad \gamma = \frac{5}{6}.$$

5.2. Stokes-line smoothing

As usual we truncate the algebraic series for θ and q optimally

$$\theta = \sum_{n=0}^{N-3}\epsilon^{n/3}\theta_n + R_N, \qquad q = \sum_{n=0}^{N-3}\epsilon^{n/3}q_n + S_N,$$

where R_N satisfies

$$\epsilon\frac{\mathrm{d}R_N}{\mathrm{d}t}+\frac{1}{3}\frac{\mathrm{d}u}{\mathrm{d}t}\left(R_N-\epsilon^{\frac{N}{3}}\theta_N-\epsilon^{\frac{N-1}{3}}\theta_{N-1}-\epsilon^{\frac{N-2}{3}}\theta_{N-2}\right)$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta\mathrm{e}^{2\pi\mathrm{i}k/3}u^{1/3}\right)\left(\epsilon^{2/3}R_N-\epsilon^{\frac{N+2}{3}}\theta_N-\epsilon^{\frac{N+1}{3}}\theta_{N-1}\right)+\epsilon f R_N=\cdots$$

where

$$f=\left(\left(\frac{1+\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_0}\sin\theta_0}{\cos^3\theta_0}\right)\frac{t\theta_3}{1+t^2}+\mathrm{i}\frac{\mathrm{d}\theta_0}{\mathrm{d}t}\right)$$

and the omitted terms are lower order as $N\to\infty$ and $\epsilon\to 0$.

The presence of $\mathrm{e}^{\beta n^{1/3}}$ in the behaviour of θ_n moves the optimal truncation point from $N\sim r/\epsilon$ to $N\sim r/\epsilon-(r/\epsilon)^{1/3}\mathfrak{Re}(\beta\mathrm{e}^{-2\mathrm{i}\theta/3})$ where $u\mathrm{e}^{2\pi\mathrm{i}k}=r\mathrm{e}^{\mathrm{i}\theta}$. This moves the Stokes line by a distance $\epsilon^{2/3}$. However, since the width of the Stokes line is $\epsilon^{1/2}$, the leading order Stokes smoothing can be observed without this extra complication. Writing $R_N=A(u)\mathrm{e}^{-\frac{u}{3\epsilon}}\mathrm{e}^{\frac{\mathrm{e}^{2\pi\mathrm{i}k/3}\beta u^{1/3}}{\epsilon^{1/3}}}$ gives

$$\epsilon\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}A}{\mathrm{d}u}+\epsilon f A=\frac{1}{3}\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{e}^{-\frac{\mathrm{e}^{2\pi\mathrm{i}k/3}\beta u^{1/3}}{\epsilon^{1/3}}}\mathrm{e}^{\frac{u}{3\epsilon}}\left(\epsilon^{\frac{N}{3}}\theta_N+\epsilon^{\frac{N-1}{3}}\theta_{N-1}+\epsilon^{\frac{N-2}{3}}\theta_{N-2}\right)+\cdots.$$

With $N=r/\epsilon+\alpha$, where α is $O(1)$ as $\epsilon\to 0$, we find

$$\epsilon^{\frac{N}{3}}\theta_N\sim\frac{G_k}{r^{1/2}\epsilon^{\gamma}\mathrm{e}^{\mathrm{i}\theta(\gamma+1/2)}}\exp\left[\left(\frac{r}{3\epsilon}+\frac{\alpha}{3}\right)\left(\log\left(\frac{r}{\epsilon}+\alpha\right)\frac{\epsilon}{r\mathrm{e}^{\mathrm{i}\theta}}-1+3\beta\frac{\epsilon^{2/3}}{r^{2/3}}\right)+\frac{r\mathrm{e}^{\mathrm{i}\theta}}{3\epsilon}-\beta\frac{r^{1/3}\mathrm{e}^{\mathrm{i}\theta/3}}{\epsilon^{1/3}}\right]$$

which is exponentially small except near the Stokes line $\theta=0$. Setting $\theta=\epsilon^{1/2}\bar{\theta}$ gives

$$\epsilon^{\frac{N}{3}}\theta_N\sim\frac{G_k}{r^{1/2}\epsilon^{\gamma}}\mathrm{e}^{-\frac{r\bar{\theta}^2}{6}}.$$

Writing

$$\frac{\mathrm{d}}{\mathrm{d}u}=-\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}}{r}\frac{\mathrm{d}}{\mathrm{d}\theta}$$

the local equation in the vicinity of the Stokes line is therefore

$$\frac{\mathrm{d}A}{\mathrm{d}\bar{\theta}}=\frac{\mathrm{i}G_k}{\epsilon^{\gamma+1/2}}\mathrm{e}^{-\frac{r\bar{\theta}^2}{6}},$$

so that the jump in A across the Stokes line is given by

$$[A]=\frac{\sqrt{6\pi}\,\mathrm{i}G_k}{\epsilon^{\gamma+1/2}}. \tag{124}$$

The relevant Stokes lines are that for $k=0$ on which $\arg(u)=0$, which lies down the imaginary axis from $t=\mathrm{i}$, and the corresponding Stokes line up the imaginary axis from $t=-\mathrm{i}$, and shown in Figure 4(c). Stokes lines corresponding to other values of k do not intersect the real axis. Thus there will be a solution to the boundary value problem only if $\Theta_0=0$. To determine Θ_0 we need to match this outer solution with an inner solution in the vicinity of $t=\mathrm{i}$.

6. Matching with the inner region

Defining the inner variable z by

$$t = \mathrm{i} + \epsilon^{2/3} z, \tag{125}$$

we find

$$\cos\theta_0 \sim \frac{\mathrm{e}^{-3\mathrm{i}\pi/4}}{2^{1/2}\epsilon^{1/3}z^{1/2}}, \tag{126}$$

$$\sin\theta_0 \sim -\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{2^{1/2}\epsilon^{1/3}z^{1/2}}. \tag{127}$$

$$\log q - \mathrm{i}\theta \sim -\log 2 - \epsilon^{2/3}\left(\frac{a}{4} + \frac{\mathrm{i}z}{2}\right). \tag{128}$$

$$\theta_0 \sim \frac{\mathrm{i}}{2}\log z - \mathrm{i}\log\left(\frac{2^{1/2}\mathrm{e}^{-3\mathrm{i}\pi/4}}{\epsilon^{1/3}}\right). \tag{129}$$

We define the inner solution ϕ by

$$q = \frac{\mathrm{e}^{-3\mathrm{i}\pi/4}}{2^{1/2}\epsilon^{1/3}z^{1/2}}\phi, \tag{130}$$

giving, to leading order,

$$-2\sqrt{2}\,\mathrm{e}^{-3\mathrm{i}\pi/4}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi_0}{z^{1/2}}\right) = 1 - \frac{1}{\phi_0^2} - \frac{\mathrm{i}a}{2z}. \tag{131}$$

The change of variables

$$\zeta = \frac{\mathrm{i}z}{2}, \tag{132}$$

gives

$$-\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\phi_0}{\zeta^{1/2}}\right) = 1 - \frac{1}{\phi_0^2} + \frac{a}{4\zeta}. \tag{133}$$

As before, we are interested in the outer limit of this inner expansion. Expanding as $\zeta \to \infty$ we find

$$\phi \sim \sum_{n=0}^{\infty}\frac{A_n}{\zeta^{n/2}} = \sum_{n=0}^{\infty} A_n\left(\frac{2}{\mathrm{i}z}\right)^{n/2}, \tag{134}$$

where the A_n satisfy the recurrence relation

$$\frac{1}{2}\sum_{m=0}^{n-3}\sum_{k=0}^{n-m-3}(k+1)A_k A_m A_{n-m-k-3} = \sum_{m=0}^{n} A_{n-m}A_m + \frac{a}{4}\sum_{m=2}^{n} A_{n-m}A_{m-2}, \quad n \geq 3, \tag{135}$$

with

$$A_0 = 1, \qquad A_1 = 0, \qquad A_2 = -\frac{a}{8}, \qquad A_3 = \frac{1}{4}. \tag{136}$$

It is easy to check that the large n behaviour of A_n is

$$A_n \sim \frac{\Lambda n^{n/3+\gamma} \mathrm{e}^{-n/3+\beta_0 n^{1/3}}}{4^{n/3}},$$

where $\gamma = 5/6$ and $\beta_0 = -3a/2^{5/3}$. In fact, there are three possible large n behaviours corresponding to the three roots of $4^{1/3}$ and $n^{1/3}$. Since A_n is real we have

$$A_n \sim \frac{\Lambda_1 n^{n/3+\gamma} \mathrm{e}^{-n/3+\beta_0 n^{1/3}}}{4^{n/3}} + \frac{\Lambda_2 \, \mathrm{e}^{2\mathrm{i}n\pi/3} n^{n/3+\gamma} \mathrm{e}^{-n/3+\mathrm{e}^{2\mathrm{i}\pi/3}\beta_0 n^{1/3}}}{4^{n/3}} + \frac{\bar{\Lambda}_2 \, \mathrm{e}^{-2\mathrm{i}n\pi/3} n^{n/3+\gamma} \mathrm{e}^{-n/3+\mathrm{e}^{-2\mathrm{i}\pi/3}\beta_0 n^{1/3}}}{4^{n/3}}.$$

For $a < 0$, the real root dominates, while for $a > 0$ the complex conjugate pair dominates. From the outer expansion we have as $t \to \mathrm{i}$,

$$\epsilon^{n/3} q_n \sim \epsilon^{n/3} \mathrm{i} q_0 \theta_n \sim \frac{\mathrm{i} q_0 \Theta_{k,0} \mathrm{e}^{-2\pi \mathrm{i} n k/3} n^{n/3+\gamma} \mathrm{e}^{-n/3+\mathrm{e}^{-2\pi \mathrm{i} k/3}\beta_0 n^{1/3}}}{2^{2n/3} \zeta^{n/2}},$$

where $\Theta_{k,0}$ is the inner limit of Θ_k. Matching this with the inner solution gives

$$\Theta_{0,0} = -\mathrm{i}\Lambda_1, \qquad \Theta_{1,0} = -\mathrm{i}\bar{\Lambda}_2, \qquad \Theta_{2,0} = -\mathrm{i}\Lambda_2.$$

Because the real root dominates when $a < 0$ it is easy to see that $\Theta_{0,0} \neq 0$, the Stokes line down the imaginary axis is active, and there is no solution to the boundary value problem. However, when $a > 0$, $\Theta_{0,0}$ is buried in the late terms of A_n and it is hard to tell for what values of a the condition for a solution $\Theta_{0,0} = 0$ is met. Doing so requires a super-exponential analysis of the recurrence relation, or a numerical solution of the inner problem (133), both of which are beyond the scope of the present paper. Fortunately, the key conclusion (namely that $\lambda = 1/2$ is selected by kinetic undercooling in the limit $\epsilon \to 0$) follows from the analysis above without the need for such refinements.

7. Discussion

The rôle of surface tension in selecting a discrete spectrum of Saffman-Taylor fingers has been widely analysed. Here, in part as a step towards analysing the effects of more general regularisations within a unified framework, we have studied the influence of another physically meaningful one, namely kinetic undercooling. The main conclusion (the selection of the $\lambda = 1/2$ Saffman-Taylor finger when the regularising parameter is small) is the same as for surface tension and again requires the careful tracking of exponentially small terms; this is not a coincidence, the case $\lambda = 1/2$ being special within the family of Saffman-Taylor fingers with regard to its complex plane singularities.

The beyond-all-orders problem described in Sections 3–6 is notably subtle. Firstly, the Stokes lines exhibit rather exotic behaviour at the origin (where the real line passes through them). We believe that the model problem discussed in Appendix 2 is the simplest way to elucidate such behaviour but we note that such difficulties can to a large extent be obviated in the asymptotic calculation by diverting the contour around the origin. The real line interpretation is important, however, in clarifying the role of the analyticity of the interface in the selection process (*cf.* Appendix A1.5). In particular we note the possibility of fingers which are even in y but are non-analytic at their tips, with the non-analyticity arising for small kinetic undercooling in a term which is exponentially small. Numerically it would presumably be extremely difficult to exclude such solutions, and thereby determine the required discrete spectrum of analytic interfaces; we leave the numerical treatment of this problem as an open challenge.

Secondly, as indicated in Section 6, it seems that a 'hyperasymptotic' calculation is required if correction terms to $\lambda = 1/2$ are to be constructed; to our knowledge, this is the first such example and it motivates further studies directed at developing the necessary techniques.

We conclude by emphasising the effectiveness of asymptotic techniques in identifying and describing the relevant solutions for both small and large kinetic undercooling, providing a rather complete picture of the structure of travelling wave solutions to the Hele-Shaw problem with kinetic undercooling.

Appendix 1, The Hele-Shaw problem with kinetic undercooling

A1.1. Preamble

This appendix is intended to set the beyond-all-orders analysis above into context by discussing some of the properties of

$$\begin{aligned} \nabla^2\phi &= 0, && \\ \frac{\partial\phi}{\partial y} &= 0 && \text{on} \quad y = \pm 1, \\ \phi &\sim x && \text{as} \quad x \to +\infty, \\ \phi &= cv_n = c\frac{\partial\phi}{\partial n} && \text{on} \quad x = f(y,t), \end{aligned} \tag{A1.1}$$

where f is the moving boundary (this corresponds to (1)–(3) above with slightly different scalings; n is in the inward normal direction to the fluid), for $O(1)$ values of the kinetic undercooling parameter c. We largely concentrate, for reasons which will become clear shortly, on cases where $f(y,t)$ is finite for all $|y| \le 1$, so no condition is required as $x \to -\infty$; as in [11], the problem is not then correctly specified unless the contact angles ($\tan^{-1}(\pm 1/\partial f/\partial y)$ at $y = \pm 1$) are prescribed and less than $\pi/2$ or are required to be greater than or equal to $\pi/2$; we shall focus mainly on the latter case.

It is worth noting that the Hele-Shaw problem with kinetic undercooling inherits the invariance under arbitrary changes of clock of the unregularised problem; thus, if the condition in (A1.1) as $x \to +\infty$ is generalised to $\phi \sim \dot{T}(t)x,\ \dot{T} > 0$, the change of variables $\hat{t} = T(t),\ \phi = \dot{T}(t)\hat{\phi},\ v_n = \dot{T}(t)\hat{v}_n$ enables (A1.1) to be recovered and, in particular, for travelling waves to be sought (this is not the case with surface tension, for example). The 'blowing' case ($\dot{T} < 0$) corresponds to a time-reversal ($\hat{t} = -t,\ \phi = -\hat{\phi},\ v_n = -\hat{v}_n$) of the 'sucking' one ($\dot{T} > 0$), a property which the surface-tension regularisation again lacks.

A1.2. Linear Stability

Perturbing off the base state $\phi = x - t + c,\ f = t$ by writing

$$\phi \sim x - t + c + \Phi,\ f \sim t + F$$

yields

$$\begin{aligned}
\nabla^2\Phi &= 0, \\
\frac{\partial\Phi}{\partial y} &= 0 && \text{on} \quad y = \pm 1, \\
\Phi &\to 0 && \text{as} \quad x \to +\infty, \\
\Phi &= -F + c\frac{\partial F}{\partial t}, \ \frac{\partial F}{\partial t} = \frac{\partial \Phi}{\partial x} && \text{on} \quad x = t,
\end{aligned}$$

which has solutions

$$\Phi = A_k \mathrm{e}^{-k(x-t)} \mathrm{e}^{\frac{k}{1+kc}t} \left\{ \begin{array}{c} \cos ky \\ \sin ky \end{array} \right\}, \ F = -(1+kc)A_k \mathrm{e}^{\frac{k}{1+kc}t} \left\{ \begin{array}{c} \cos ky \\ \sin ky \end{array} \right\} \tag{A1.2}$$

for suitable k, so (unlike the surface-tension regularisation) all wavenumbers are unstable; nevertheless the kinetic undercooling term provides the required moderation of the growth rate for large k.

A1.3. Travelling waves

Provided $f(y, t)$ is finite for all y, (A1.1) admits the exact travelling-wave solution

$$\phi = x - t + c, \quad f = t + g(y)$$

(the possibility of such a solution follows from the kinematic condition and it is thus applicable to most regularisations), where

$$g + c = c \Bigg/ \left(1 + \left(\frac{\mathrm{d}g}{\mathrm{d}y}\right)^2\right)^{\frac{1}{2}}.$$

This has a one-parameter family of solutions

$$g = \sqrt{c^2 - (y - y_0)^2} - c,$$

where y_0 is an arbitrary constant, and an envelope solution $g = 0$. Since kinetic undercooling does not require the interface to be analytic, so that, in particular, it may contain a corner of angle less than π (see [21]), we may thus construct the following continuum of solutions for $\pi/2$ contact angles:

$$\text{(a)} \qquad g = \sqrt{c^2 - (1 - |y|)^2} - c \quad \text{for } c \geq 1. \tag{A1.3}$$

$$\text{(b)} \qquad g = 0, \quad |y| \geq y_0, \quad g = \sqrt{c^2 - (y_0 - |y|)^2} - c, \ |y| \leq y_0, \ 0 < y_0 < \min(c, 1), \tag{A1.4}$$

where we have restricted attention to even solutions. Related solutions corresponding to higher modes are readily constructed, such as

$$g = \sqrt{c^2 - y^2} - c \quad |y| \leq \tfrac{1}{2}, \quad g = \sqrt{c^2 - (1 - |y|)^2} - c, \quad |y| \geq \tfrac{1}{2} \quad \text{for } c \geq \tfrac{1}{2} \tag{A1.5}$$

Solutions such as (A1.4) in which the interface is partially straight are expected to be unstable (*cf.* Section A1.2); this is perhaps most simply clarified by analysing the limit $c \to \infty$ whereby under the scalings

$$t = c\hat{t}, \quad f = c\hat{t} + \hat{f}/c, \quad \phi \sim c + z + \hat{\phi}_0(\hat{t})/c,$$

where $z = x - c\hat{t}$, we obtain

$$\frac{\partial \hat{f}_0}{\partial \hat{t}} = \tfrac{1}{2}\left(\frac{\partial \hat{f}_0}{\partial y}\right)^2 + \hat{f}_0 + \hat{\phi}_0(\hat{t}), \tag{A1.6}$$

in which $\hat{\phi}_0$ is determined by conservation of mass and is readily eliminated from (A1.6) by a suitable translation of $\hat{f}_0$. Thus for $c \geq 1$, we expect (A1.3) to give the stable travelling-wave profile (these being non-planar for all finite c, in keeping with Section A1.2); as c drops below one we expect a configuration of the type shown in Figure 1 to develop, except that, for the current thought experiment in which c is progressively decreased, two 'half-fingers' of gas will result adjacent to the walls, with fluid down the middle of the channel; our two scenarios are mathematically equivalent due to symmetry. In fact, since (as noted above) a contact angle greater than $\pi/2$ cannot be imposed, the family of solutions

$$g = \sqrt{c^2 - (y - y_0)^2} - c, \quad |y_0| < \min(c - 1, 1) \tag{A1.7}$$

is as acceptable as (A1.3), bifurcating to the configuration in Figure 1 as c drops below one. When $y_0 = 0$, the solution (A1.7) has contact angle $\phi = \pi - \tan^{-1}\left(\sqrt{c^2 - 1}\right)$ and the solutions for $c \geq 1$ are best characterised by this angle, rather than by λ (since $\lambda \equiv 1$ holds).

Solutions for contact angles $\phi < \pi/2$ are also readily constructed (for brevity we consider the case where the same angle is enforced at $y = \pm 1$); the analog of (A1.3) is

$$g = \sqrt{c^2 - (1 + c\cos\phi - |y|)^2} - c, \quad c \geq 1/(1 - \cos\phi)$$

so the bifurcation at which the gas ceases to occupy the full channel width occurs at increasing values of c as ϕ is decreased; this is unsurprising, since the relevant configuration corresponds closely to $\pi/2$ contact angle fingers in progressively wider channels.

A1.4. $c \to 1^-$

We now briefly examine the limit $\lambda \to 1^-$. The moving-boundary problem (A1.1) is supplemented by

$$\frac{\partial\phi}{\partial x} \to 0 \quad \text{as} \quad x \to -\infty, \quad |y| > 1 - \lambda.$$

The wavespeed is given by $U = 1/\lambda$ and we thus write

$$f = Ut + g(y).$$

The two relevant small parameters are $\varepsilon = 1 - c$ and $\delta(\varepsilon) = 1 - \lambda$; it will turn out that $\delta \ll \varepsilon$ so that $U = 1 + O(\varepsilon)$ and, correct at $O(\varepsilon)$, we can take as our outer solution (*cf.* (A1.7))

$$g \sim \sqrt{(1 - \varepsilon)^2 - y^2} - (1 - \varepsilon). \tag{A1.8}$$

The inner scalings are

$$x = Ut - (1 - \varepsilon) + \delta^{\frac{1}{2}} Z, \quad y = 1 - \delta Y, \quad \phi = \delta^{\frac{1}{2}}\Phi$$

so that (*cf.* [12])

$$\Phi \sim \Phi_0(Z) + \delta\left(-\tfrac{1}{2}\frac{\mathrm{d}^2\Phi_0}{\mathrm{d}Z^2}Y^2 + \Phi_1(Z)\right)$$

and, writing the free boundary as $Y = h(Z)$,

$$\Phi_0 = -\frac{\mathrm{d}h_0}{\mathrm{d}Z} = -\frac{\mathrm{d}}{\mathrm{d}Z}\left(h_0\frac{\mathrm{d}\Phi_0}{\mathrm{d}Z}\right).$$

Hence

$$\frac{\mathrm{d}h_0}{\mathrm{d}Z} = \left(2\left(h_0 - \log h_0 - 1\right)\right)^{\frac{1}{2}},$$

from which it follows that

$$h_0 \sim \tfrac{1}{2}Z^2 + 2\log Z + 3 - \log 2 \quad \text{as} \quad Z \to +\infty. \tag{A1.9}$$

Since (A1.8) implies that

$$h \sim \tfrac{1}{2}Z^2 + \frac{\varepsilon}{\delta},$$

matching with (A1.9) for $Z = O(\delta^{-\frac{1}{2}})$ implies

$$\varepsilon \sim \delta\log(1/\delta) \qquad \text{as} \quad \delta \to 0$$

i.e.

$$\lambda \sim 1 + \frac{(1-c)}{\log(1-c)} \qquad \text{as} \quad c \to 1^-,$$

giving the asymptotic behaviour of λ in the complementary limit to that discussed in the bulk of the paper.

A1.5. The tip of the finger

As already noted, the interface $x = f(y,t)$ in (A1.1) need not be analytic and this can have significant implications for the nature of travelling-wave solutions. Here we analyse the local behaviour at a non-analytic tip at $(s(t), 0)$ by writing $x = s(t) + z$ and, for brevity treating the case of solutions symmetric about $y = 0$,

$$\phi \sim a(t) + \dot{s}z + b(t)(z^2 - y^2) + \ldots,$$
$$f \sim s(t) - \beta(t)y^2 + F(y,t),$$

where $F = o(y^2)$; the moving-boundary conditions yield

$$c\dot{s} = a, \quad c\dot{\beta} + (2c\beta - 1)\beta\dot{s} = b,$$

with a and b being determined as part of the solution. It follows from $\phi = cv_n$ that the terms in F of interest here (namely those which arise from non-analyticities in the initial data $f(y,0)$ rather than being driven by the flow ϕ) satisfy

$$\frac{\partial F}{\partial t} + 2\beta\dot{s}y\frac{\partial F}{\partial y} = \frac{\dot{s}}{c}F; \tag{A1.10}$$

the growth rate in (A1.2) as $k \to \infty$ corresponds to (A1.10) with $\dot{s} = 1$ and with the second term negligible, as is to be expected. Hence if $f(y,0) = F_0(y), s(0) = 0$, it follows from (A1.10) that

$$F = e^{\frac{s}{c}} F_0\left(y \big/ e^{2\int_0^t \beta(t')\dot{s}(t')\mathrm{d}t'}\right), \tag{A1.11}$$

so that, in particular, non-analyticities of the form

$$F_0(y) = A|y|^m, \quad m > 2, \tag{A1.12}$$

grow for sufficiently small m but decay for large m; if $s \sim Ut,\ \beta \to \beta_\infty$ as $t \to \infty$ then

$$F \sim B e^{(1-2m\beta_\infty c)Ut/c}|y|^m \quad \text{as} \quad t \to \infty, \tag{A1.13}$$

so the exponent m for neutral stability, namely

$$m_c = 1/2\beta_\infty c, \tag{A1.14}$$

increases as c decreases (as is to be expected from the regularising role of kinetic undercooling). We conclude from (A1.11)), (A1.13) that families of travelling-wave solutions to (A1.1) should exist which are non-analytic at the tip (the non-analyticity being related to the local behaviour (*i.e.*, β_∞) via (A1.14)), but that such travelling waves can arise only if the initial data contain a suitable term of the form (A1.12) with $m = m_c$; in other words, it seems likely that travelling wave solutions of arbitrary velocity exist for arbitrary λ, but that they have non-analytic interfaces (this not being permitted in the case of surface tension) and can arise only from very special initial data (being unstable). The waves we have selected above in the limits $c \to 0^+$ and $c \to 1^-$ are those with analytic free boundaries and we expect these to arise generically for large time from wide classes of initial data, with the analysis above making explicit some of the subtleties of the selection process.

Appendix 2, Behaviour at the origin – linear paradigm

In this appendix, we analyse a simple linear model problem, namely

$$-\varepsilon t \frac{\mathrm{d}\phi}{\mathrm{d}t} + \phi = \frac{1}{t^2+1}, \tag{A2.1}$$

to clarify the (remarkably complicated) structure of the Stokes line close to $t = 0$, a complication which can be circumvented (as we have done above) by taking a path of integration which avoids the origin.

We specify ϕ uniquely by requiring that

$$\phi \to 0 \quad \text{as} \quad t \to -\infty$$

and that ϕ be real along the real line (the complementary function

$$\phi = At^{1/\varepsilon},$$

is exponentially small for $|t| < 1$, and this latter condition relates to the placing of branch cuts through the origin; both $t = 0$ and $t = \infty$ are regular singular points of (A2.1) and other branch cuts emanate from the logarithmic singularities in ϕ at $t = \pm\mathrm{i}$). The exact solution to (A2.1) is then

$$\phi = \frac{1}{\varepsilon}(-t)^{\frac{1}{\varepsilon}} \int_{-\infty}^{t} \frac{1}{(-t')^{\frac{1}{\varepsilon}+1}(t'^2+1)}\,\mathrm{d}t',$$

and repeated integration by parts gives that

$$\phi = \sum_{n=0}^{N} \frac{(-1)^n(-t)^{2n}}{1-2\varepsilon n} - \frac{(-1)^N}{\varepsilon}(-t)^{\frac{1}{\varepsilon}} \int_{-\infty}^{t} \frac{1}{(-t')^{\frac{1}{\varepsilon}-2N-1}(1+t'^2)}\,\mathrm{d}t', \tag{A2.2}$$

so choosing $N \gg 1$ such that

$$2N < 1/\varepsilon < 2N+2$$

(for brevity, we avoid discussion of the case in which $1/\varepsilon$ is an even integer) we may deduce that in $|t| < 1$ we can write ϕ as the superposition of

$$\alpha(\varepsilon)(-t)^{\frac{1}{\varepsilon}}, \tag{A2.3}$$

where

$$\alpha = -\frac{(-1)^N}{\varepsilon} \int_{-\infty}^{0} \frac{1}{(-t')^{\frac{1}{\varepsilon}-2N-1}(1+t'^2)}\,\mathrm{d}t' = -\frac{\pi}{2\varepsilon\sin(\pi/2\varepsilon)}, \tag{A2.4}$$

and a convergent Taylor expansion (*cf.* (A2.2)). We note that the quantity (A2.3) corresponds very closely to (A1.13) with $m = m_c$ (with t and ε here corresponding to y and c, respectively, there). The small t expansion of ϕ is thus perfectly innocuous; the small ε expansion, and the associated Stokes phenomenon, is not, however. Since (A2.1) is invariant under $t \to -t$, we may deduce that an expression of the form

$$\alpha\left((-t)^{\frac{1}{\varepsilon}} - t^{\frac{1}{\varepsilon}}\right)$$

will be present as $t \to +\infty$; for this to be real, we must split the first term into two equal parts and we place the branch cut of one up the positive imaginary axis and that of the other down the negative one, leaving

$$\alpha(\cos(\pi/\varepsilon) - 1)t^{\frac{1}{\varepsilon}} = \pi\,\sin(\pi/2\varepsilon)t^{\frac{1}{\varepsilon}}/\varepsilon \tag{A2.5}$$

as the term we expect to be turned on in part via the Stokes phenomenon.

We place all the branch cuts along the imaginary axis. The jump $[\phi]_-^+$ across them (where $+$ denotes positive, and $-$ negative, real part of t) satisfies, writing $t = \mathrm{i}\tau$,

$$-\varepsilon\tau\frac{\mathrm{d}}{\mathrm{d}\tau}[\phi]_-^+ + [\phi]_-^+ = 0$$

away from $\tau = \pm 1$. Since

$$\phi \sim \frac{1}{2\varepsilon}\log(1-|\tau|) \quad \text{as} \quad \tau \to \pm 1$$

we have for $\tau > 1$ a contribution

$$\frac{\pi\mathrm{i}}{\varepsilon}\left(\frac{t}{\mathrm{i}}\right)^{\frac{1}{\varepsilon}} \tag{A2.6}$$

to $[\phi]_-^+$ and for $\tau < -1$ a contribution

$$-\frac{\pi \mathrm{i}}{\varepsilon}\left(\frac{t}{-\mathrm{i}}\right)^{\frac{1}{\varepsilon}}. \tag{A2.7}$$

Similarly, splitting (A2.3) into two equal parts, as outlined above, we have a contribution of

$$\alpha \mathrm{i} \sin(\pi/\varepsilon) t^{\frac{1}{\varepsilon}} = -\pi \mathrm{i} \cos(\pi/2\varepsilon) t^{\frac{1}{\varepsilon}}/\varepsilon$$

to $[\phi]_-^+$ on the positive imaginary axis and of

$$-\alpha \mathrm{i} \sin(\pi/\varepsilon) t^{\frac{1}{\varepsilon}} = \pi \mathrm{i} \cos(\pi/2\varepsilon) t^{\frac{1}{\varepsilon}}/\varepsilon$$

on the negative one. Summing the relevant expressions, the branch cut information thus tells us that

$$[\phi]_-^+ = \pi \sin(\pi/2\varepsilon) t^{\frac{1}{\varepsilon}}/\varepsilon \qquad \text{for} \quad |\tau| > 1,$$

precisely in keeping with (A2.5), and that

$$[\phi]_-^+ = -\pi \mathrm{i} \cos(\pi/2\varepsilon) t^{\frac{1}{\varepsilon}}/\varepsilon \quad \text{for} \quad 0 < \tau < 1, \tag{A2.8}$$

$$[\phi]_-^+ = \pi \mathrm{i} \cos(\pi/2\varepsilon) t^{\frac{1}{\varepsilon}}/\varepsilon \qquad \text{for} \quad -1 < \tau < 0, \tag{A2.9}$$

so (prejudging the Stokes line analysis, the above descriptions of the branch cuts providing rather complete information for this paradigm) it follows that the Stokes lines $0 < \tau < 1$ and $-1 < \tau < 0$ should switch on the quantities (A2.6) and (A2.7) respectively.

Having established these exact properties of (A2.1), we now turn to an analysis of the limit $\varepsilon \to 0^+$, in which

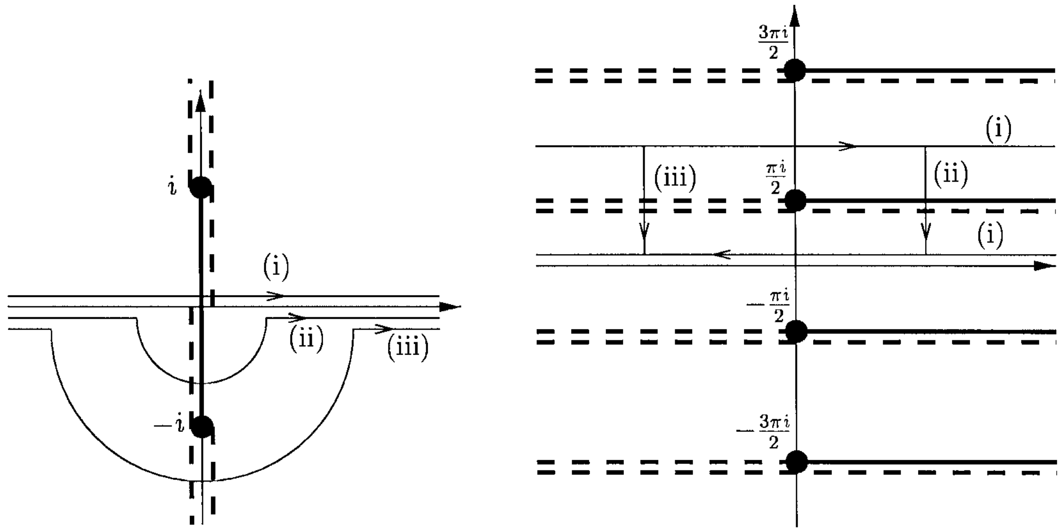
$$\phi_0 = \frac{1}{t^2+1}$$

is the leading-order outer solution. There are inner regions in the complex plane at $t = \pm\mathrm{i} + O(\varepsilon)$ which generate Stokes lines running along the imaginary axis to the origin. To perform the analysis of these Stokes lines it is convenient to transform independent variables, whereby we write $t = \mathrm{e}^{-s}$ to give

$$\varepsilon \frac{\mathrm{d}\phi}{\mathrm{d}s} + \phi = \tfrac{1}{2}(1 + \tanh s)$$

which is equivalent to a problem analysed for quite different reasons in [22] (though we align the branch cuts in a different way here). The complex s plane contains multiple copies of the complex t plane and Figure 6 gives sample integration paths (i)–(iii) in each.

On path (iii), we turn on (A2.5) via the two branch cuts, as described above. On path (ii) we cross a single branch cut and a single Stokes line; the analysis of [22] shows that this Stokes line switches on precisely the required quantity (A2.7) via the usual error function smoothing. The discussion of the path (i), however, is more delicate and brings us to the crux of the matter. This path apparently crosses no Stokes lines; however, Figure 6(b) is misleading in the sense that the thickness of the Stokes lines grows as $\sqrt{\varepsilon \mathfrak{Re}(s)}$, so for $\mathfrak{Re}(s) = O(1/\varepsilon)$ all the Stokes lines merge (equivalently, given the $2\pi\mathrm{i}$ periodicity in the s plane, the two with $\mathfrak{Im}(s) = \pm\pi\mathrm{i}/2$, corresponding to those in Figure 6(a), merge) and the integration path (i) runs through this merged array thereby turning on the requisite quantity (A2.7). As shown in [22], this infinite array of Stokes lines leads to a Stokes switching that is significantly more complicated than the usual error function. Via this merging of Stokes lines, it is also shown there that, along the right-ward component of (i), a quantity



(a) (b)

Figure 6. Sample integration paths in (a) the t plane and (b) the s plane. Dark solid lines are Stokes lines, dashed lines are branch cuts.

$$-\pi\,\mathrm{e}^{-(s-\pi\mathrm{i})/\varepsilon}\big/2\varepsilon\sin(\pi/2\varepsilon)$$

emerges as $\mathfrak{Re}(s) \to +\infty$, consistent with (A2.3)–(A2.4); moreover, this term forms part of a convergent series corresponding to (A2.2); in other words, as the various Stokes line in Figure 6(b) blur the Stokes phenomenon (associated with divergent expansions) dissipates as the convergent large $\mathfrak{Re}(s)$ series emerges in a rather delicate way from the small ε asymptotic expansion. (To clarify this point, we note that the large n ansatz for ϕ_n of the factorial over power type described in [17], for example, is not valid for sufficiently large $\mathfrak{Re}(s)$ (specifically, $\mathfrak{Re}(s) = O(n^{\frac{1}{2}})$ in the current example); a related point is that the limits $\varepsilon \to 0$ and $s \to \infty$ evidently do not commute). Equivalently, in the t plane the Stokes lines disappear in the neighbouring of origin by passing through a region (corresponding to $\mathfrak{Re}(s) = O(1/\varepsilon)$) in which $|t|$ is exponentially small but $\arg(t)$ varies over the full range.

In summary, we stress the following points.

(i) The nature of the Stokes switching across the imaginary axis, as described by an $\varepsilon \to 0$ asymptotic expansion, is non-trivial for $|t|$ exponentially small, with the convergent $t \to 0$ series (A2.2) providing a much simpler representation of the solution in this neighbourhood.

(ii) The Stokes lines emanating from $t = \pm\mathrm{i}$ each terminate at the origin.

(iii) Taking an integration path in the complex plane which avoids the origin (as above) yields valid results whilst avoiding many of the difficulties alluded to above.

Appendix 3: Explicit expressions for *u* and *F*

The expression (56) may be integrated explicitly to give

$$
\begin{aligned}
u = {} & 2\,\tan^{-1} x - (1-b^2)^{1/2}\,(1+b)\,\tan^{-1}\left(\frac{(1-b)^{1/2}\,x}{(1+b)^{1/2}}\right) \\
& -(1-b^2)^{1/2}\,(1-b)\,\tan^{-1}\left(\frac{(1+b)^{1/2}\,x}{(1-b)^{1/2}}\right) + b^2\,\log(1-x) - b^2\,\log(1+x)
\end{aligned}
$$

where

$$
b = \sqrt{1+\alpha}, \qquad x = \sqrt{\frac{b+\mathrm{i}\,t}{b-\mathrm{i}\,t}}.
$$

The expression (64) for $F(t)$ may be simplified to

$$
\begin{aligned}
F(t) = {} & \frac{2\sqrt{\alpha}}{\pi(1+\sqrt{1+\alpha})} - \frac{2((1+\alpha)^{1/2}+\mathrm{i}t)^{1/2}}{\pi((1+\alpha)^{1/2}-\mathrm{i}t)^{1/2}} + \frac{4}{\pi}\log\left(\frac{(1+\alpha+t^2)^{1/2}+\sqrt{1+\alpha}}{(-\mathrm{i}t)(\sqrt{\alpha}+\sqrt{1+\alpha})}\right) \\
& - \frac{\sqrt{1+\alpha}(1+\mathrm{i}t)}{2(1+\sqrt{1+\alpha})(\sqrt{1+\alpha}-\mathrm{i}t)} - 2\log(-\mathrm{i}t) + \frac{5}{4}\log\left(\frac{\mathrm{i}t+\sqrt{1+\alpha}}{\sqrt{1+\alpha}-1}\right) \\
& + \frac{3}{4}\log\left(\frac{-\mathrm{i}t+\sqrt{1+\alpha}}{1+\sqrt{1+\alpha}}\right) + \frac{\sqrt{1+\alpha}-1}{4(\sqrt{1+\alpha}+1)} - \frac{\sqrt{1+\alpha}+\mathrm{i}t}{4(\sqrt{1+\alpha}-\mathrm{i}t)} \\
& - \frac{3}{4}\log\left(\frac{(\sqrt{1+\alpha}+\mathrm{i}t)(\sqrt{1+\alpha}+1)}{(\sqrt{1+\alpha}-\mathrm{i}t)(\sqrt{1+\alpha}-1)}\right) \\
& - \frac{2}{\pi}\int_{\frac{(\sqrt{1+\alpha}-1)^{1/2}}{(\sqrt{1+\alpha}+1)^{1/2}}}^{\frac{(\sqrt{1+\alpha}+\mathrm{i}t)^{1/2}}{(\sqrt{1+\alpha}-\mathrm{i}t)^{1/2}}} \frac{(3+\tau^2)\tan^{-1}\tau}{\tau}\,\mathrm{d}\tau.
\end{aligned}
\tag{A3.10}
$$

It follows that, as $t \to \mathrm{i}\sqrt{1+\alpha}$,

$$
\begin{aligned}
F \sim {} & \frac{1}{2}\log(t-\mathrm{i}\sqrt{1+\alpha}) + \frac{\mathrm{i}\pi}{4} + \frac{2\sqrt{\alpha}}{\pi(1+\sqrt{1+\alpha})} - \frac{4}{\pi}\log\left(\sqrt{\alpha}+\sqrt{1+\alpha}\right) \\
& + \frac{1}{2}\frac{(\sqrt{1+\alpha}-1)}{(\sqrt{1+\alpha}+1)} - \frac{1}{4}\log(1+\alpha) + \frac{3}{2}\log 2 \\
& - \frac{1}{2}\log(\sqrt{1+\alpha}-1) - \frac{3}{2}\log(\sqrt{1+\alpha}+1) \\
& - \frac{2}{\pi}\int_{\frac{(\sqrt{1+\alpha}-1)^{1/2}}{(\sqrt{1+\alpha}+1)^{1/2}}}^{0} \frac{(3+\tau^2)\tan^{-1}\tau}{\tau}\,\mathrm{d}\tau + \cdots .
\end{aligned}
\tag{A3.11}
$$

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